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Bounds on the covering radius of some classes of codes over $\mathbb{R}$

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Abstract: In this paper, the covering radius of codes over $\mathbb{R} = \mathbb{Z}_2 R^*$, where $R^* = \mathbb{Z}_2 + v\mathbb{Z}_2, v^2 = v$ with different weight are discussed. The block repetition codes over $\mathbb{R}$ is defined and the covering radius for block repetition codes, simplex code of α-type and β-type in $\mathbb{R}$ are obtained.

Keywords: Additive codes, covering radius, gray map, Lee weight, Euclidean weight, simplex codes.

MSC: 11T71, 94B05, 11H71.

1. Introduction

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic. Recently, there has been substantial interest in the class of additive codes. In [1,2], Delsarte contributes to the algebraic theory of association scheme where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme.

The covering radius is an important geometric parameter of codes. It not only indicates the maximum error correcting capability of codes, but also relates to some practical problems such as the data compression and transmission. Studying of the covering radius of codes has attracted many coding scientists for almost 30 years. The covering radius of linear codes over binary finite fields was studied in [3].

Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [4–7]. Enormous results were made available on the simplex codes over finite fields and finite rings. A few of them are [8–10]. In [11], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2$ where $u^2 = 0$ with respect to different distance, and they explained the covering radius of various repetition codes, Simplex Codes (Type α and β over R.) The above results motivate us to work in this area.

In this paper, the simplex codes of types α and β over $\mathbb{R}$ are obtained. At this juncture, the meaning of constructing new codes is to concatenate binary and quaternary simplex codes of types α and β. These results in the simplex codes over $\mathbb{R}$, which contain the corresponding binary and R^* codes as subclasses. The rest of this paper is organized as follows

In Section 2, some properties related to additive codes over $\mathbb{R} = \mathbb{Z}_2 R^*$, where $R^* = \{0, 1, v, 1+v\}, v^2 = v$ are given. The covering radius of codes over $\mathbb{R}$ is considered in Section 3. In Section 4, the block repetition codes over $\mathbb{R}$ is defined and find their covering radius. The construction of simplex codes over $\mathbb{R}$ of types α and β and the covering radius of these codes is also considered in Sections 5.

2. Preliminaries

Throughout this paper, $\mathbb{R} = \mathbb{Z}_2 R^*$, where $R^* = \{0, 1, v, 1+v\}, v^2 = v$. In this section, some preliminary results are given based on [5] and [7]. A non empty set C is a $\mathbb{R}$-additive code if it is a subgroup of $\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^{\lambda} \times R^{*^{\mu}}$ for some λ and μ. C is of type $2^{\lambda} R^{*^{\mu}}$ as a group. It follows that it has $\mid C \mid = 2^{\lambda+2\mu}$ codewords, and the number of order two codewords in C is $\mid C \mid = 2^{\lambda+\mu}$.

Consider the following extension of the Gray map

$$\phi : \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}} \to \mathbb{Z}_2^{n}, \text{ with } n = \gamma + 2\delta,$$

given by

$$\phi\,(u, w) = (u, \phi\,(w_1)\,, \cdots \phi\,(w_\delta))\,, \forall\ u\ \in\ \mathbb{Z}_2^{\gamma}, \forall\ (w_1, \cdots, w_\delta)\ \in\ R^{*^{\delta}},$$

where

$$\phi : R^* \to \mathbb{Z}_2^2,$$

is the Gray map given by $\phi\,(0) = (0,\ 0)\,, \phi\,(1) = (0,\ 1)\,, \phi\,(v) = (1,\ 1)\,,$ and $\phi\,(1+v) = (1,\ 0)\,.$ Then the binary image of a $\mathbb{R}$-additive code under the extended Gray map is called a $\mathbb{R}$-linear code of length $n = \gamma + 2\delta$.

The Hamming weight of u, denoted by $wt_H\,(u)$ and $wt_L\,(w)$ and $wt_E\,(w)$ the Lee and Euclidean weights of w respectively, where $u \in \mathbb{Z}_2^{\gamma}$ and $w \in R^{*^{\delta}}$ are defined as

$$wt_E\,(v_i) = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i = 1 \text{ or } (1+v) \\ 4 & \text{if } v_i = v \end{cases}$$

and

$$wt_L\,(v_i) = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i = 1 \text{ or } (1+v) \\ 2 & \text{if } v_i = v \end{cases}$$

The Lee weight of x is defined as $wt_L\,(x) = wt_H\,(u) + wt_L\,(w)\,,$ where $x = (u,\ w)\ \in\ \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$, and $u = (u_1,\ \cdots,\ u_\gamma)\ \epsilon\ \mathbb{Z}_2^{\gamma}$ and $w = (w_1,\ \cdots,\ w_\delta)\ \epsilon\ R^{*^{\delta}}$ and the Euclidean weight of x is defined as $wt_E\,(x) = wt_H\,(u) + wt_E\,(w)\,.$ The Gray map defined above is an isometry which transforms the Lee distance defined over $\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$ to the Hamming distance defined over $\mathbb{Z}_2^{n}$, with $n = \gamma + 2\delta$.

3. The covering radius of codes over $\mathbb{R}$

The covering radius of a code C over $\mathbb{R}$ is introduced in this section. The covering radius of a code C is the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$. Hence, the covering radius of a code C over $\mathbb{R}$, with respect to the Lee and Euclidean distances is given by

$$r_L\,(C) = \max_{u\epsilon \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}} \left\{ \min_{c\epsilon C} d_L\,(u, c) \right\} \text{ and } r_E\,(C) = \max_{u\epsilon \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}} \left\{ \min_{c\epsilon C} d_E\,(u, c) \right\}, \tag{1}$$

respectively.

The following result, for codes over $\mathbb{Z}_4$, given by Aoki *et al.* in [12] is also valid for codes over $\mathbb{R}$. Its proof follows from the definition of the covering radius and the fact that the map ϕ is a weight preserving map.

Proposition 1. *Let C be a code over $\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$ and $\phi\,(C)$ be the Gray map image of C. Then $r_L\,(C) = r_H\,(\phi\,(C))\,.$*

The following result is useful for determining the covering radius of codes over rings. This is a generalization of the result in [13] for codes over finite fields.

Proposition 2. *If C_0 and C_1 are codes over $\mathbb{R}$, of lengths n_0 and n_1, minimum distance d_0 and d_1, and generated by matrices G_0 and G_1, respectively, and if C is the code generated by*

$$G = \left[\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right],$$

then $r_d(C) \le r_d(C_0) + r_d(C_1)$, and the covering radius of the concatenation of C_0 and C_1, denoted C_c, satisfies

$$r_d(C_c) \ge r_d(C_0) + r_d(C_1)$$

for all distances d over $\mathbb{R}$.

4. The covering radius of the block repetition codes over $\mathbb{R}$

In order to determine the covering radius of simplex codes of types α and β over $\mathbb{R}$, the some classes of block codes over $\mathbb{R}$ is defind and the approach in [14] is used to obtain the covering radius.

The block repetition code C^n over $\mathbb{R}$ is a $\mathbb{R}$-additive code of length $n = \sum_{j=1}^{7} n_j$ with generator matrix $G =$

$$\left(\overbrace{0101\cdots01}^{n_1}\overbrace{0v0v\cdots0v}^{n_2}\overbrace{01+v01+v\cdots01+v}^{n_3}\overbrace{1010\cdots10}^{n_4}\overbrace{1111\cdots11}^{n_5}\overbrace{1v1v\cdots1v}^{n_6}\overbrace{11+v11+v\cdots11+v}^{n_7}\right).$$

If, for a fixed $1 \le i \le 7$. For all $1 \le j \ne i \le 7, n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the seven basic repetition codes are given below. That,

1. $C_1 = \{(00\cdots00), (01\cdots01), (0v\cdots0v), (01+v\cdots01+v)\}$, is an additive code of length $n = n_1$ generated to $G_1 = [0101\cdots01]$.
2. $C_2 = \{(00\cdots00), (0v\cdots0v)\}$, is an additive code of length $n = n_2$ generated to $G_2 = [0v0v\cdots0v]$.
3. $C_3 = \{(00\cdots00), (01\cdots01), (0v\cdots0v), (01+v\cdots01+v)\}$, is an additive code of length $n = n_3$ generated to $G_3 = [01+v01+v\cdots01+v]$,
4. $C_4 = \{(00\cdots00), (10\cdots10)\}$, is an additive code of length $n = n_4$ generated to $G_4 = [1010\cdots10]$,
5. $C_5 = \{(00\cdots00), (01\cdots01), (0v\cdots0v), (01+v\cdots01+v), (10\cdots10), (11\cdots11), (1v\cdots1v), (11+v\cdots11+v)\}$, is an additive code of length $n = n_5$ generated to $G_5 = [1111\cdots11]$,
6. $C_6 = \{(00\cdots00), (0v\cdots0v), (10\cdots10), (1v\cdots1v)\}$, is an additive code of length $n = n_6$ generated to $G_6 = [1v1v\cdots1v]$,
7. $C_7 = \{(00\cdots00), (01\cdots01), (0v\cdots0v), (01+v\cdots01+v), (10\cdots10), (11\cdots11), (1v\cdots1v), (11+v\cdots11+v)\}$, is an additive code of length $n = n_7$ generated to $G_7 = [11+v11+v\cdots11+v]$.

The following theorems provide the covering radius of C_j, for $1 \le j \le 7$.

Theorem 3. *The covering radius of $C_j, 1 \le j \le 7$, with respect to the Euclidean weight is given by*

1. $\frac{3n}{4} \le r_E(C_1) = r_E(C_3) \le 2n$
2. $n \le r_E(C_2) \le 3n$
3. $\frac{n}{4} \le r_E(C_4) \le 4n$,
4. $n \le r_E(C_5) = r_E(C_7) \le 2n$,
5. $\frac{5n}{4} \le r_E(C_6) \le \frac{5n}{2}$.

Proof. For $c \in C_j, 1 \le j \le 7$, let $t_i(c), 0 \le i \le 7$ denote the number of occurrences of symbol i in the codeword c. Considering 1 to 5, that

$$r_E(C_j) = \max_{x\in\mathbb{R}^n}\{d_E(x, C_j); 1 \le j \le 7\}.$$

Let $x \in \mathbb{R}^n$. If x is given by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^{7} t_j = n$, then

$d_E(x, \overline{00}) = n - t_0 + 3t_2 + t_5 + 4t_6 + t_7, d_E(x, \overline{01}) = n - t_1 + 3t_3 + t_4 + t_6 + 4t_7,$
$d_E(x, \overline{0v}) = n - t_2 + 3t_0 + 4t_4 + t_5 + t_7, d_E(x, \overline{01+v}) = n - t_3 + 3t_1 + t_4 + 4t_5 + t_6,$
$d_E(x, \overline{10}) = n - t_4 + t_1 + 4t_2 + t_3 + 3t_6, d_E(x, \overline{11}) = n - t_5 + t_0 + t_2 + 4t_3 + 3t_7,$
$d_E(x, \overline{1v}) = n - t_6 + 4t_0 + t_1 + t_3 + 3t_4, d_E(x, \overline{11+v}) = n - t_7 + t_0 + 4t_1 + t_2 + 3t_5.$

Therefore, $d_E(x, C_1) = d_E(x, C_3) = \min\{(n - t_0 + 3t_2 + t_5 + 4t_6 + t_7), (n - t_1 + 3t_3 + t_4 + t_6 + 4t_7), (n - t_2 + 3t_0 + 4t_4 + t_5 + t_7), (n - t_3 + 3t_1 + t_4 + 4t_5 + t_6)\} \le \frac{4n+2(t_0+t_1+t_2+t_3+t_4+t_5+t_6+t_7)+2(t_4+t_5+t_6+t_7)}{4} \le 2n$ and hence

$$r_E(C_1) = r_E(C_3) \le 2n.$$

If $x = \left(\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{01\cdots01}^{\frac{n}{4}}\overbrace{0v\cdots0v}^{\frac{n}{4}}\overbrace{01+v\cdots01+v}^{\frac{n}{4}} \right) \in \mathbb{R}^n$, then

$$d_E\left(x,\overline{00}\right) = d_E\left(x,\overline{01}\right) = d_E\left(x,\overline{0u}\right) = d_E\left(x,\overline{01+u}\right) = \frac{n}{8} + 4\left(\frac{n}{8}\right) + \frac{n}{8} = \frac{3n}{4}$$

and $r_E\left(C_1\right) = r_E\left(C_3\right) \geq \frac{3n}{4}$. Thus, $\frac{3n}{4} \leq r_E\left(C_1\right) = r_E\left(C_3\right) \leq 2n$.

$$\begin{aligned} d_E\left(x,C_2\right) &= \min\left\{\left(n-t_0+3t_2+t_5+4t_6+t_7\right),\left(n-t_2+3t_0+4t_4+t_5+t_7\right)\right\} \\ &\leq \frac{2n+2\left(t_0+t_2+t_5+t_7\right)+4\left(t_4+t_6\right)}{2} \leq 3n. \end{aligned}$$

Thus $r_E\left(C_2\right) \leq 3n$.

If $x = \left(\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{0v\cdots0v}^{\frac{n}{2}} \right) \in \mathbb{R}^n$, then

$$d_E\left(x,\overline{00}\right) = d_E\left(x,\overline{0v}\right) = 4\left(\frac{n}{4}\right) = n.$$

Thus $r_E\left(C_2\right) \geq n$, and so, $n \leq r_E\left(C_2\right) \leq 3n$.

$$\begin{aligned} d_E\left(x,C_4\right) &= \min\left\{\left(n-t_0+3t_2+t_5+4t_6+t_7\right),\left(n-t_4+t_1+4t_2+t_3+3t_6\right)\right\} \\ &\leq \frac{2n-t_0+t_1+7t_2+t_3-t_4+t_5+7t_6+t_7}{2} \leq 4n \end{aligned}$$

and hence $r_E\left(C_4\right) \leq 4n$.

If $x = \left(\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{10\cdots10}^{\frac{n}{2}} \right) \in \mathbb{R}^n$, then

$$d_E\left(x,\overline{00}\right) = d_E\left(x,\overline{10}\right) = \frac{n}{4}.$$

Thus $r_E\left(C_4\right) \geq \frac{n}{4}$, and so $\frac{n}{4} \leq r_E\left(C_4\right) \leq 4n$.
$d_E(x,C_5) = d_E(x,C_7) = \min\{(n-t_0+3t_2+t_5+4t_6+t_7),(n-t_1+3t_3+t_4+t_6+4t_7),(n-t_2+3t_0+4t_4+t_5+t_7),(n-t_3+3t_1+t_4+4t_5+t_6),(n-t_4+t_1+4t_2+t_3+3t_6),(n-t_5+t_0+t_2+4t_3+3t_7),(n-t_6+4t_0+t_1+t_3+3t_4),(n-t_7+t_0+4t_1+t_2+3t_5)\} \leq 2n$, then $r_E(C_5) = r_E(C_7) \leq 2n$.

If $x = \left(\overbrace{00\cdots00}^{\frac{n}{8}}\overbrace{01\cdots01}^{\frac{n}{8}}\overbrace{0v\cdots0v}^{\frac{n}{8}}\overbrace{01+v\cdots01+v}^{\frac{n}{8}}\overbrace{10\cdots10}^{\frac{n}{8}}\overbrace{11\cdots11}^{\frac{n}{8}}\overbrace{1v\cdots1v}^{\frac{n}{8}}\overbrace{11+v\cdots11+v}^{\frac{n}{8}} \right) \in \mathbb{R}^n$, then

$d_E\left(x,\overline{00}\right) = d_E\left(x,\overline{01}\right) = d_E\left(x,\overline{0v}\right) = d_E\left(x,\overline{01+v}\right) = d_E\left(x,\overline{10}\right) = d_E\left(x,\overline{11}\right) = d_E\left(x,\overline{1v}\right) = d_E\left(x,\overline{11+v}\right) = \frac{n}{16} + 4\left(\frac{n}{16}\right) + \frac{n}{16} + \frac{n}{16} + \frac{n}{8} + \frac{n}{16} + 4\left(\frac{n}{16}\right) + \frac{n}{8} = n.$

Thus $r_E\left(C_5\right) = r_E\left(C_7\right) \geq n$, and so $n \leq r_E\left(C_5\right) = r_E\left(C_7\right) \leq 2n$.

$d_E(x,C_6) = \min\{(n-t_0+3t_2+t_5+4t_6+t_7),(n-t_2+3t_0+4t_4+t_5+t_7),(n-t_4+t_1+4t_2+t_3+3t_6),(n-t_6+4t_0+t_1+t_3+3t_4)\} \leq \frac{4n+2n+4n}{4}$, so then $r_E\left(C_6\right) \leq \frac{5n}{2}$.

If $x = \left(\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{0v\cdots0v}^{\frac{n}{4}}\overbrace{10\cdots10}^{\frac{n}{4}}\overbrace{1v\cdots1v}^{\frac{n}{4}} \right) \epsilon\, \mathbb{R}^n$,

then

$d_E\left(x,\overline{00}\right) = d_E\left(x,\overline{0v}\right) = d_E\left(x,\overline{10}\right) = d_E\left(x,\overline{1v}\right) = 4\left(\frac{n}{8}\right) + \frac{n}{8} + \frac{n}{8} + 4\left(\frac{n}{8}\right) = \frac{5n}{4}.$

Thus $r_E\left(C_6\right) \geq \frac{5n}{4}$, and so $\frac{5n}{4} \leq r_E\left(C_6\right) \leq \frac{5n}{2}$. $\square$

Theorem 4. *The Covering radius of $C_j, 1 \leq j \leq 7$, with respect to the Lee weight is given by*

1. $\frac{n}{2} \leq r_L(C_1) = r_E(C_3) \leq 2n$,
2. $\frac{n}{2} \leq r_L(C_2) \leq 2n$,
3. $\frac{n}{4} \leq r_L(C_4) \leq 2n$,
4. $\frac{3n}{4} \leq r_L(C_5) = r_L(C_7) \leq \frac{3n}{2}$,
5. $\frac{3n}{4} \leq r_L(C_6) \leq \frac{3n}{2}$.

Proof. For $c \in C_j, 1 \leq j \leq 7$, let $t_i(c), 0 \leq i \leq 7$ denote the number of occurrences of symbol i in the codeword c. Considering 1 to 5, that

$$r_L(C_j) = \max_{x \in \mathbb{R}^n} \{d_L(x, C_j); 1 \leq j \leq 7\}.$$

Let $x \in \mathbb{R}^n$. If x is given by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^{7} t_j = n$, then
$d_L(x, \overline{00}) = n - t_0 + t_2 + t_5 + 2t_6 + t_7, d_L(x, \overline{01}) = n - t_1 + t_3 + t_4 + t_6 + 2t_7,$
$d_L(x, \overline{0v}) = n - t_2 + t_0 + 2t_4 + t_5 + t_7, d_L(x, \overline{01+v}) = n - t_3 + t_1 + t_4 + 2t_5 + t_6,$
$d_L(x, \overline{10}) = n - t_4 + t_1 + 2t_2 + t_3 + t_6, d_L(x, \overline{11}) = n - t_5 + t_0 + t_2 + 2t_3 + t_7,$
$d_L(x, \overline{1v}) = n - t_6 + 2t_0 + t_1 + t_3 + t_4, d_L(x, \overline{11+v}) = n - t_7 + t_0 + 2t_1 + t_2 + t_5.$
Therefore, $d_L(x, C_1) = d_L(x, C_3) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_1 + t_3 + t_4 + t_6 + 2t_7), (n - t_2 + t_0 + 2t_4 + t_5 + t_7), (n - t_3 + t_1 + t_4 + 2t_5 + t_6)\} \leq \frac{4n + 4(t_4 + t_5 + t_6 + t_7)}{4} \leq 2n$, then $r_L(C_1) = r_L(C_3) \leq 2n$.

If $x = \left(\overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{01 \cdots 01}^{\frac{n}{4}} \overbrace{0v \cdots 0v}^{\frac{n}{4}} \overbrace{01+v \cdots 01+v}^{\frac{n}{4}}\right) \in \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = d_L(x, \overline{0v}) = d_L(x, \overline{01+v}) = \frac{n}{8} + 2\left(\frac{n}{8}\right) + \frac{n}{8} = \frac{n}{2}$. Thus $r_L(C_1) = r_L(C_3) \geq \frac{n}{2}$, and so $\frac{n}{2} \leq r_L(C_1) = r_L(C_3) \leq 2n$.

$d_L(x, C_2) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_2 + t_0 + 2t_4 + t_5 + t_7)\} \leq \frac{2n + 2(t_4 + t_5 + t_6 + t_7)}{2} \leq 2n$. Then $r_L(C_2) \leq 2n$.

If $x = \left(\overbrace{00 \cdots 00}^{\frac{n}{2}} \overbrace{0v \cdots 0v}^{\frac{n}{2}}\right) \in \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{0v}) = 2\left(\frac{n}{4}\right) = \frac{n}{2}$. Thus $r_L(C_2) \geq \frac{n}{2}$ and so $\frac{n}{2} \leq r_L(C_2) \leq 2n$.

$d_L(x, C_4) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_4 + t_1 + 2t_2 + t_3 + t_6)\} \leq \frac{2n + 2n}{2} \leq 2n$, so then $r_L(C_4) \leq 2n$.

If $x = \left(\overbrace{00 \cdots 00}^{\frac{n}{2}} \overbrace{10 \cdots 10}^{\frac{n}{2}}\right) \in \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{10}) = \left(\frac{n}{4}\right) = \frac{n}{4}$. Thus $r_L(C_4) \geq \frac{n}{4}$ and hence $\frac{n}{4} \leq r_L(C_4) \leq 2n$.

$d_L(x, C_5) = d_L(x, C_7) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_1 + t_3 + t_4 + t_6 + 2t_7), (n - t_2 + t_0 + 2t_4 + t_5 + t_7), (n - t_3 + t_1 + t_4 + 2t_5 + t_6), (n - t_4 + t_1 + 2t_2 + t_3 + t_6), (n - t_5 + t_0 + t_2 + 2t_3 + t_7), (n - t_6 + 2t_0 + t_1 + t_3 + t_4), (n - t_7 + t_0 + 2t_1 + t_2 + t_5)\} \leq \frac{8n - n + 5n}{8} \leq \frac{3n}{2}$, so then $r_L(C_5) = r_L(C_7) \leq \frac{3n}{2}$.

If $x = \left(\overbrace{00 \cdots 00}^{\frac{n}{8}} \overbrace{01 \cdots 01}^{\frac{n}{8}} \overbrace{0v \cdots 0v}^{\frac{n}{8}} \overbrace{01+v \cdots 01+v}^{\frac{n}{8}} \overbrace{10 \cdots 10}^{\frac{n}{8}} \overbrace{11 \cdots 11}^{\frac{n}{8}} \overbrace{1v \cdots 1v}^{\frac{n}{8}} \overbrace{11+v \cdots 11+v}^{\frac{n}{8}}\right) \in \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) = d_L(x, \overline{0v}) = d_L(x, \overline{01+v}) = d_L(x, \overline{10}) = d_L(x, \overline{11}) = d_L(x, \overline{1v}) = d_L(x, \overline{11+v}) = \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} + \frac{n}{16} + \frac{n}{16} + \frac{n}{16} + \frac{n}{16} + 2\left(\frac{n}{16}\right) + \frac{n}{16} + \frac{n}{16} = \frac{3n}{4}$. Thus $r_L(C_5) = r_L(C_7) \geq \frac{3n}{4}$, and so $\frac{3n}{4} \leq r_L(C_5) = r_L(C_7) \leq \frac{3n}{2}$. $d_L(x, C_6) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_2 + t_0 + 2t_4 + t_5 + t_7), (n - t_4 + t_1 + 2t_2 + t_3 + t_6), (n - t_6 + 2t_0 + t_1 + t_3 + t_4)\} \leq \frac{4n + 2n}{4} \leq \frac{3n}{2}$, so then $r_L(C_6) \leq \frac{3n}{2}$.

If $x = \left(\overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{0v \cdots 0v}^{\frac{n}{4}} \overbrace{10 \cdots 10}^{\frac{n}{4}} \overbrace{1v \cdots 1v}^{\frac{n}{4}}\right) \in \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{0v}) = d_L(x, \overline{10}) = d_L(x, \overline{1v}) = 2\left(\frac{n}{8}\right) + \frac{n}{8} + \frac{n}{8} + 2\left(\frac{n}{8}\right) = \frac{3n}{4}$. Thus $r_L(C_6) \geq \frac{3n}{4}$, and hence $\frac{3n}{4} \leq r_L(C_6) \leq \frac{3n}{2}$. □

Theorem 5. *The covering radius of the block repetition code C^n have the following properties;*

$$r_E\left(C^n\right) \le \frac{1}{2}\left[5\left(n_1+n_3+n_6\right)+3n_2+9n_4\right]+2\left(n_5+n_7\right),$$

and if $n_1 = \cdots = n_7 = n$

$$r_L\left(C^{7n}\right) = 2n.$$

Proof. By Proposition 2, Theorem 3 and Theorem 4, let $x = x_1x_2x_3x_4x_5x_6x_7 \in \mathbb{R}^n$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7), (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7)$, $(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7), (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7)$, $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7), (f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7)$, $(g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7)$, respectively such that $n_1 = \sum\limits_{j=0}^{7} a_j$, $n_2 = \sum\limits_{j=0}^{7} b_j$, $n_3 = \sum\limits_{j=0}^{7} c_j$, $n_4 = \sum\limits_{j=0}^{7} d_j$, $n_5 = \sum\limits_{j=0}^{7} e_j$, $n_6 = \sum\limits_{j=0}^{7} f_j$, $n_7 = \sum\limits_{j=0}^{7} g_j$. Then $d_E\left(x, \overline{00}\right) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_0 + 3e_2 + e_5 + 4e_6 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_0 + 3g_2 + g_5 + 4g_6 + g_7$, where $\overline{00} = \overbrace{00\cdots00}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{00\cdots00}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{00\cdots00}^{n_5}\overbrace{00\cdots00}^{n_6}\overbrace{00\cdots00}^{n_7}$, is the first vector of C^n, where $n = n_1$.

$d_E\left(x, \overline{y_1}\right) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + c_4 + 4c_5 + c_6 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_5 + e_0 + e_2 + 4e_3 + 3e_7 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_7 + g_0 + 4g_1 + g_2 + 3g_5$, where $\overline{y_1} = \overbrace{01\cdots01}^{n_1}\overbrace{0v\cdots0v}^{n_2}\overbrace{01+v\cdots01+v}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{11\cdots11}^{n_5}\overbrace{1v\cdots1v}^{n_6}\overbrace{11+v\cdots11+v}^{n_7}$, is the second vector of C^n, where $n = n_2$.

$d_E\left(x, \overline{y_2}\right) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + c_4 + 4c_5 + c_6 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_1 + 3e_3 + e_4 + e_6 + 4e_7 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_3 + 3g_1 + g_4 + 4g_5 + g_6$, where $\overline{y_2} = \overbrace{01\cdots01}^{n_1}\overbrace{0v\cdots0v}^{n_2}\overbrace{01+v\cdots01+v}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{01\cdots01}^{n_5}\overbrace{0v\cdots0v}^{n_6}\overbrace{01+v\cdots01+v}^{n_7}$, is the third vector of C^n, where $n = n_3$.

$d_E\left(x, \overline{y_3}\right) = n_1 - a_2 + 3a_0 + 4a_4 + a_5 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_2 + 3e_0 + 4e_4 + e_5 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_2 + 3g_0 + 4g_4 + g_5 + g_7$, where $\overline{y_3} = \overbrace{0v\cdots0v}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{0v\cdots0v}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{0v\cdots0v}^{n_5}\overbrace{00\cdots00}^{n_6}\overbrace{0v\cdots0v}^{n_7}$, is the fourth vector of C^n, where $n = n_4$.

$d_E\left(x, \overline{y_4}\right) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_3 + 3e_1 + e_4 + 4e_5 + e_6 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_1 + 3g_3 + g_4 + g_6 + 4g_7$, where $\overline{y_4} = \overbrace{01+v\cdots01+v}^{n_1}\overbrace{0v\cdots0v}^{n_2}\overbrace{01\cdots01}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{01+v\cdots01+v}^{n_5}\overbrace{0v\cdots0v}^{n_6}$ $\overbrace{01+v\cdots01+v}^{n_7}$, is the fifth vector of C^n, where $n = n_5$.

$d_E\left(x, \overline{y_5}\right) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_4 + e_1 + 4e_2 + e_3 + 3e_6 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_4 + g_1 + 4g_2 + g_3 + 3g_6$, where $\overline{y_5} = \overbrace{00\cdots00}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{00\cdots00}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{10\cdots10}^{n_5}\overbrace{10\cdots10}^{n_6}\overbrace{10\cdots10}^{n_7}$, is the sixth vector of C^n, where $n = n_6$.

$d_E\left(x, \overline{y_6}\right) = n_1 - a_2 + 3a_0 + 4a_4 + a_5 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_6 + 4e_0 + e_1 + e_3 + 3e_4 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_6 + 4g_0 + g_1 + g_3 + 3g_4$, where $\overline{y_6} = \overbrace{0v\cdots0v}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{0v\cdots0v}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{1v\cdots1v}^{n_5}\overbrace{10\cdots10}^{n_6}\overbrace{1v\cdots1v}^{n_7}$, is the seventh vector of C^n, where $n = n_7$.

$d_E\left(x, \overline{y_7}\right) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_7 + e_0 + 4e_1 + e_2 + 3e_5 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_5 + g_0 + g_2 + 4g_3 + 3g_7$, where $\overline{y_7} = \overbrace{01+v\cdots01+v}^{n_1}\overbrace{0v\cdots0v}^{n_2}\overbrace{01\cdots01}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{11+v\cdots11+v}^{n_5}\overbrace{1v\cdots1v}^{n_6}\overbrace{11\cdots11}^{n_7}$, is the eighth vector of C^n, where $n = n_8$.

Thus

$$r_E\left(C^n\right) \leq \frac{8n_1+4\left(a_0+a_1+a_2+a_3\right)+12\left(a_4+a_5+a_6+a_7\right)+8n_2+8\left(b_0+b_2+b_5+b_7\right)+16\left(b_4+b_6\right)}{8}$$
$$+\frac{8n_3+4\left(c_0+c_1+c_2+c_3\right)+12\left(c_4+c_5+c_6+c_7\right)+8n_4-4\left(d_0+d_4\right)+4\left(d_1+d_3+d_5+d_7\right)+28\left(d_2+d_6\right)}{8}$$
$$+\frac{8n_5+8\left(e_0+e_1+e_2+e_3+e_4+e_5+e_6+e_7\right)+8n_6+12\left(f_0+f_2+f_4+f_6\right)+4\left(f_1+f_3+f_5+f_7\right)}{8}$$
$$+\frac{8n_7+8\left(g_0+g_1+g_2+g_3+g_4+g_5+g_6+g_7\right)}{8}.$$

Hence

$$r_E\left(C^n\right) \leq \tfrac{1}{2}\left[5\left(n_1+n_3+n_6\right)+3n_2+9n_4\right]+2\left(n_5+n_7\right).$$

For the second part, that $\phi\left(C^{7n}\right)$ is the set given by
$\{000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000,$
$001\cdots001011\cdots011010\cdots010100\cdots100101\cdots101111\cdots111110\cdots110,$
$001\cdots001011\cdots011010\cdots010000\cdots000001\cdots001011\cdots011010\cdots010,$
$011\cdots011000\cdots000011\cdots011000\cdots000011\cdots011000\cdots000011\cdots011,$
$010\cdots010011\cdots011001\cdots001000\cdots000010\cdots010011\cdots011001\cdots001,$
$000\cdots000000\cdots000000\cdots000100\cdots100100\cdots100100\cdots100100\cdots100,$
$011\cdots011000\cdots000011\cdots011100\cdots100111\cdots111100\cdots100111\cdots111,$
$010\cdots010011\cdots011001\cdots001100\cdots100110\cdots110111\cdots111101\cdots101\}.$
Then $r_L\left(c^{7n}\right)=r\left(\phi\left(C^{7n}\right)\right)=2n.$ □

5. Simplex codes of types α and β

In this section, consider the construction of simplex codes of types α and β over $\mathbb{R}$. Let $m^{\alpha}_{2,k}$ be the generator matrix of $S^{\alpha}_{2,k}$, the binary simplex code of type α, defined as

$$\left[\begin{array}{c|c} 00\cdots0 & 11\cdots1 \\ \hline m^{\alpha}_{2,k-1} & m^{\alpha}_{2,k-1}\end{array}\right], \text{ for } k\geq 2,$$

where

$$m^{\alpha}_{2,1}=[0,1].$$

In [8], the simplex codes $S^{\alpha}_{4,k}$ of type α over R^* were defined. The generator matrix $G^{\alpha}_{R^*,k}$ of $S^{\alpha}_{R^*,k}$ is

$$\left[\begin{array}{c|c|c|c} 00\cdots0 & 11\cdots1 & vv\cdots v & 1+v1+v\cdots1+v \\ \hline G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1}\end{array}\right], \text{ for } k\geq 2,$$

where

$$G^{\alpha}_{R^*,1}=[0\;1\;v\;1+v].$$

The generator matrix of S^{α}_{k}, the simplex code of type α over $\mathbb{R}$ is dfined, as the concatenation of 2^{2k} copies of the generator matrix of $S^{\alpha}_{2,k}$ and 2^k copies of the generator matrix of $S^{\alpha}_{R^*,k}$ given by

$$\Theta^{\alpha}_{k}=\left[\begin{array}{c|c|c|c|c|c|c|c} m^{\alpha}_{2,k} & m^{\alpha}_{2,k} & \cdots & m^{\alpha}_{2,k} & G^{\alpha}_{R^*,k} & G^{\alpha}_{R^*,k} & \cdots & G^{\alpha}_{R^*,k}\end{array}\right], \tag{2}$$

for $k\geq 1$.

The standard form of Θ^{α}_{k} of the generator matrix of S^{α}_{k} is given by

$$\Theta^{\alpha}_{k}=\left[\begin{array}{c|c|c|c} 00\;00\cdots00 & 01\;01\cdots01 & \cdots & 11+v1\;1+v\cdots11+v \\ \hline \Theta^{\alpha}_{k-1} & \Theta^{\alpha}_{k-1} & \cdots & \Theta^{\alpha}_{k-1}\end{array}\right], \text{ for } k\geq 2,$$

where

$$\Theta_1^{\alpha} = [00\ 01\ 0v\ 01+v\ 10\ 11\ 1v\ 11+v]$$

The length of the simplex code of type α over $\mathbb{R}$ is equal to 2^{3k+1}, and the number of codewords is equal to $2^{k_0}R^{*k_1}$ for some k_0 and k_1. In the case where $k=1$ with $k_0=0$ and $k_1=1$, that all of the codewords of the simplex code S_1^{α} are generated by Θ_1^{α} and are given by

$$\begin{gathered} 00\ 00\ 00\ 00\ 00\ 00\ 00\ 00 \\ 00\ 01\ 0v\ 01+v\ 10\ 11\ 1v\ 11+v \\ 00\ 0v\ 00\ 0v\ 00\ 0v\ 00\ 0v \\ 00\ 01+v\ 0v\ 01\ 10\ 11+v\ 1v\ 11 \end{gathered}$$

The type β simplex code S_k^{β} is a punctured version of S_k^{α}. The number of codewords is $2^{k_0}R^{*k_1}$ for some k_0 and k_1 and its length is $2^k\left(2^{k-2}+1\right)\left(2^k-1\right)$.

The generator matrix of S_k^{β} is the concatenation of 2^k copies of the generator matrix of $S_{2,k}^{\beta}$ and 2^{k-1} copies of the generator matrix of $S_{R^*,k}^{\beta}$ given by

$$\Theta_k^{\beta} = \left[\begin{array}{c|c|c|c|c|c|c|c} m_{2,k}^{\beta} & m_{2,k}^{\beta} & \cdots & m_{2,k}^{\beta} & G_{R^*,k}^{\beta} & G_{R^*,k}^{\beta} & \cdots & G_{R^*,k}^{\beta} \end{array}\right], for\ k \geq 2, \tag{3}$$

where $m_{2,k}^{\beta}$ is the generator matrix of the binary simplex code of type β is given by

$$\left[\begin{array}{c|c} 11\cdots 1 & 00\cdots 0 \\ \hline m_{2,k-1}^{\alpha} & m_{2,k-1}^{\beta} \end{array}\right], \text{ for } k \geq 3,$$

with

$$m_{2,2}^{\beta} = \left[\begin{array}{c|c} 11 & 0 \\ \hline 01 & 1 \end{array}\right],$$

and $G_{R^*,k}^{\beta}$ is a generator matrix of the simplex code over R^* of type β defined as

$$\left[\begin{array}{c|c|c} 11\cdots 1 & 00\cdots 0 & vv\cdots v \\ \hline G_{R^*,k-1}^{\alpha} & G_{R^*,k-1}^{\beta} & G_{R^*,k-1}^{\beta} \end{array}\right], \text{ for } k \geq 3,$$

with

$$G_{R^*,2}^{\beta} = \left[\begin{array}{c|c|c} 1111 & 0 & v \\ \hline 01v1+v & 1 & 1 \end{array}\right].$$

5.1. The Covering radius of the simplex codes of types α and β

The following theorems provide upper bounds on the covering radius of simplex codes over $\mathbb{R}$ with respect to the Lee and Euclidean weights.

Theorem 6. *The covering radius of the $\mathbb{R}$-simplex codes of type α are upper bounded as follows:*

$$r_L\left(S_k^{\alpha}\right) \leq 3.2^{3k-1} \text{ and } r_E\left(S_k^{\alpha}\right) \leq 2^k\left(\frac{7.2^{2k}-1}{3}\right).$$

Proof. In $\mathbb{R}$-Simplex codes of type α have a Lee weight equal to 2^{3k} or 3.2^{3k-1}. Hence, from (2), Proposition 2 and Theorem 5, we have

$$\begin{aligned}
r_L\left(S_k^{\alpha}\right) &\leq r_L\left(2^{2k}S^{\alpha}_{2,\,k}\right)+r_L\left(2^{k}S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}r_L\left(S^{\alpha}_{2,\,k}\right)+2^{k}r_L\left(S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}r_H\left(S^{\alpha}_{2,\,k}\right)+2^{k}r_L\left(S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}\left(2^{k-1}\right)+2^{k}\left[\left(3.2^{2(k-1)}+3.2^{2(k-2)}+\cdots+3.2^{2.1}\right)+r_L\left(S^{\alpha}_{R^*,\,1}\right)\right]\\
&\leq 2^{3k-1}+2^{k}\left[\left(2^{2k}-1\right)+1\right]\\
&\leq 2^{3k-1}+2^{k}.2^{2k}\\
&\leq 2^{3k-1}+2^{3k}\leq 3.2^{3k-1}.
\end{aligned}$$

Thus $r_L\left(S_k^{\alpha}\right)\leq 3.2^{3k-1}$. Similar arguments using (2), Proposition 2 and Theorem 5 give that

$$\begin{aligned}
r_E\left(S_k^{\alpha}\right) &\leq r_E\left(2^{2k}S^{\alpha}_{2,\,k}\right)+r_E\left(2^{k}S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}r_E\left(S^{\alpha}_{2,\,k}\right)+2^{k}r_E\left(S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}r_H\left(S^{\alpha}_{2,\,k}\right)+2^{k}r_E\left(S^{\alpha}_{R^*,\,k}\right)\\
&\leq 2^{2k}.2^{k-1}+2^{k}\left(\frac{11\left(2^{2k}-1\right)+9}{6}\right)\\
&\leq 2^{k}\left[2^{2k-1}+\left(\frac{11\left(2^{2k}-1\right)+9}{6}\right)\right]\\
&\leq 2^{k}\left(\frac{7.2^{2k}-1}{3}\right).
\end{aligned}$$

□

Theorem 7. *The covering radius of the* $\mathbb{R}$*-Simplex codes of type* β *are given by*
(i) $r_L\left(S_k^{\beta}\right)\leq 2^{k-1}\left[\left(2^{k-1}+1\right)\left(2^{k}-1\right)-2\right]$,
(ii) $r_E\left(S_k^{\beta}\right)\leq 2^{k-1}\left(\frac{14.2^{2k}-449}{6}\right)$.

Proof. From (1), (3), Proposition 2 and Theorem 5, we have

$$\begin{aligned}
r_L\left(S_k^{\beta}\right) &\leq r_L\left(2^{k}S^{\beta}_{2,\,k}\right)+r_L\left(2^{k-1}S^{\beta}_{R^*,\,k}\right)\\
&\leq 2^{k}r_L\left(S^{\beta}_{2,\,k}\right)+2^{k-1}r_L\left(S^{\beta}_{R^*,\,k}\right)\\
&\leq 2^{k}r_H\left(S^{\beta}_{2,\,k}\right)+2^{k-1}r_L\left(S^{\beta}_{R^*,\,k}\right)\\
&\leq 2^{k}(\frac{2^{k}-1}{)}2+2^{k-1}\left[2^{k-1}\left(2^{k}-1\right)-2\right]\\
&\leq 2^{k-1}\left(2^{k}-1\right)+2^{k-1}\left[2^{k-1}\left(2^{k}-1\right)-2\right]\\
&\leq 2^{k-1}\left[\left(2^{k-1}+1\right)\left(2^{k}-1\right)-2\right].
\end{aligned}$$

Similar arguments using (2), Proposition 2 and Theorem 5 give that

$$\begin{aligned}
r_E\left(S_k^\beta\right) &\leq r_E\left(2^k S_{2,\,k}^\beta\right) + r_E\left(2^{k-1} S_{R^*,\,k}^\beta\right) \\
&\leq 2^k r_E\left(S_{2,\,k}^\beta\right) + 2^{k-1} r_E\left(S_{R^*,\,k}^\beta\right) \\
&\leq 2^k r_H\left(S_{2,\,k}^\beta\right) + 2^{k-1} r_E\left(S_{R^*,\,k}^\beta\right) \\
&\leq 2^k\left(\frac{2^k-1}{2}\right) + 2^{k-1}\left[2^k\left(2^{k+1}-1\right) + \frac{1}{3}\left(2^{2k}-1\right) - \frac{147}{2}\right] \left(\text{since} r_E\left(S_2^\beta\right) \leq 25\right) \\
&\leq 2^{k-1}\left(2^k-1\right) + 2^{k-1}\left[2^k\left(2^{k+1}-1\right) + \frac{1}{3}\left(2^{2k}-1\right) - \frac{147}{2}\right] \\
&\leq 2^{k-1}\left[2^k\left(2^{k+1}-1\right) + \frac{1}{3}\left(2^{2k}-1\right) + \left(2^k-1\right) - \frac{147}{2}\right] \\
&\leq 2^{k-1}\left(\frac{14.2^{2k}-449}{6}\right).
\end{aligned}$$

□

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