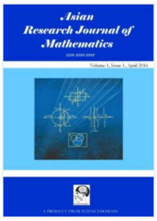

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The Number of Representations of a Positive Integer as a Sum of Four Triangular Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper we evaluate the number of representations of $n(\in \mathbb{N})$ as a sum of four triangular numbers by simple method based on Jacobi's theta functions defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}.$$

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1 Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}_0^+$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, nonnegative rational numbers, and complex numbers respectively. For $a_1, \cdots, a_4 \in \mathbb{Q}_0^+$ and $n \in \mathbb{N}_0$, we set

$$R_4(a_1, \cdots, a_4; n) := \text{card}\{(x_1, \cdots, x_4) \in \mathbb{Z}^4 | n = a_1x_1^2 + \cdots + a_4x_4^2\} \tag{1.1}$$

and

$$\begin{aligned} &N_4(a_1, \cdots, a_4; n) \\ &:= \text{card}\{(x_1, \cdots, x_4) \in \mathbb{N}_0^4 | n = a_1x_1(x_1 + 1) + \cdots + a_4x_4(x_4 + 1)\}. \end{aligned} \tag{1.2}$$

Clearly, we note that

$$R_4(a_1, \cdots, a_4; 0) = 1 \qquad \text{and} \qquad N_4(a_1, \cdots, a_4; 0) = 1.$$

If l of $a_1, \cdots, a_4$ are equal, say

$$a_i = a_{i+1} = \cdots = a_{i+l-1} = a,$$

then we indicate this in $R_4(a_1, \cdots, a_4; n)$ by writing a^l for $a_1, a_{i+1}, \cdots, a_{i+l-1}$ and it also applies to $N_4(a_1, \cdots, a_4; n)$. For $k \in \mathbb{N}$ the sum of divisors function $\sigma_k(n)$ is defined by

$$\sigma_k(n) := \begin{cases} \sum\limits_{\substack{d\in\mathbb{N} \\ d|n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \notin \mathbb{N}. \end{cases}$$

In addition we need the function

$$K_2(n) := \frac{1}{2} \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ x\equiv 1 \pmod 2 \\ n=x^2+4y^2}} (-1)^{\frac{x-1}{2}} x, \qquad n \equiv 1 \pmod 2. \tag{1.3}$$

Moreover we note that $x^2 + 4y^2 \equiv 0$ or $1 \pmod 4$ for $(x, y) \in \mathbb{Z}^2$ so that

$$K_2(n) = 0, \qquad \text{if } n \equiv 3 \pmod 4. \tag{1.4}$$

The authors and M. F. Lemire [1] have proved formulae for $R_4(1^i, 4^{4-i}; n)$ for $i \in \{1, 2, 3, 4\}$ and all $n \in \mathbb{N}$ in terms of $\sigma_1(n)$, $\sigma_1(\frac{n}{2})$, $\sigma_1(\frac{n}{4})$, $\sigma_1(\frac{n}{8})$, and $\sigma_1(\frac{n}{16})$.

Proposition 1.1. *(See Alaca et al. [1, p. 284–286]) Let $n \in \mathbb{N}$. Then*

(a)

$$R_4(1^4; n) = 8\sigma_1(n) - 32\sigma_1(\frac{n}{4}),$$

(b)

$$\begin{aligned} &R_4(1^3, 4; n) \\ &= \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma_1(n) - 20\sigma_1(\frac{n}{4}) + 24\sigma_1(\frac{n}{8}) - 32\sigma_1(\frac{n}{16}), \end{aligned}$$

(c)

$$R_4(1^2, 4^2; n) = \left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma_1(n) - 2\sigma_1(\frac{n}{2}) + 8\sigma_1(\frac{n}{8}) - 32\sigma_1(\frac{n}{16}),$$

(d)

$$R_4(1, 4^3; n) = \left(1 + \left(\frac{-4}{n}\right)\right)\sigma_1(n) - 3\sigma_1(\frac{n}{2}) + 10\sigma_1(\frac{n}{4}) - 32\sigma_1(\frac{n}{16}),$$

where $\left(\frac{-4}{n}\right)$ *is the Legendre-Jacobi-Kronecker symbol for discriminant* -4*, that is,*

$$\left(\frac{-4}{n}\right) := \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then we require

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{1.5}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}. \tag{1.6}$$

From (1.1) and (1.5) we see that

$$\sum_{n=0}^{\infty} R_4(a_1, a_2, a_3, a_4; n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \tag{1.7}$$

similarly by (1.2) and (1.6) we know that

$$\sum_{n=0}^{\infty} N_4(a_1, a_2, a_3, a_4; n)q^n = \psi(q^{a_1})\psi(q^{a_2})\psi(q^{a_3})\psi(q^{a_4}). \tag{1.8}$$

In this paper we use the Jacobi's theta function to obtain the following psi functions relations :

Theorem 1.1. *Let* $q \in \mathbb{C}$ *with* $|q| < 1$*. Then we have*

(a)

$$\psi(q)\psi^3(q^2) = \frac{1}{8}\sum_{n=0}^{\infty}\left(2\sum_{d|(8n+7)}\frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right)q^n,$$

(b)

$$\psi^2(q)\psi^2(q^2) = \frac{1}{4}\sum_{n=0}^{\infty}\sigma_1(4n+3)q^n,$$

(c)

$$\psi^3(q)\psi(q^2) = \frac{1}{8}\sum_{n=0}^{\infty}\left(\sum_{d|(8n+5)}\frac{8n+5}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+5)} d\left(\frac{8}{d}\right)\right)q^n.$$

Theorem 1.2. *Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have*

(a)

$$\psi(q)\psi^3(q^4) = \frac{1}{32}\sum_{n=0}^{\infty}\left(\sigma_1(8n+13) - 3K_2(8n+13)\right)q^n,$$

(b)

$$\psi^2(q)\psi^2(q^4) = \frac{1}{8}\sum_{n=0}^{\infty}\left(\sigma_1(4n+5) + (-1)^n K_2(4n+5)\right)q^n,$$

(c)

$$\psi^3(q)\psi(q^4) = \frac{1}{8}\sum_{n=0}^{\infty}\sigma_1(8n+7)q^n.$$

Therefore the above results enables us to deduce $N_4(a_1, \cdots, a_4; n)$ as :

Theorem 1.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$N_4(\frac{1}{2}, 1^3; n) = \frac{1}{8}\left(2\sum_{d|(8n+7)}\frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right),$$

(b)

$$N_4(\left(\frac{1}{2}\right)^2, 1^2; n) = \frac{1}{4}\sigma_1(4n+3),$$

(c)

$$N_4(\left(\frac{1}{2}\right)^3, 1; n) = \frac{1}{8}\left(\sum_{d|(8n+5)}\frac{8n+5}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+5)} d\left(\frac{8}{d}\right)\right),$$

(d)

$$N_4(\frac{1}{2}, 2^3; n) = \frac{1}{32}\left(\sigma_1(8n+13) - 3K_2(8n+13)\right),$$

(e)

$$N_4(\left(\frac{1}{2}\right)^2, 2^2; n) = \frac{1}{8}\left(\sigma_1(4n+5) + (-1)^n K_2(4n+5)\right),$$

(f)

$$N_4(\left(\frac{1}{2}\right)^3, 2; n) = \frac{1}{8}\sigma_1(8n+7).$$

2 Proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3

Proposition 2.1. *(See Alaca et al. [2, Theorem 4.3], Alaca et al. [3], Williams [4, p. 239–241]) Let $n \in \mathbb{N}$. Then we have*

(a)

$$R_4(1^3, 2; n) = 8 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - 2 \sum_{d|n} d \left(\frac{8}{d}\right),$$

(b)

$$R_4(1^2, 2^2; n) = 4\sigma_1(n) - 4\sigma_1(\frac{n}{2}) + 8\sigma_1(\frac{n}{4}) - 32\sigma_1(\frac{n}{8}),$$

(c)

$$R_4(1, 2^3; n) = 4 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - 2 \sum_{d|n} d \left(\frac{8}{d}\right),$$

(d)

$$R_4(1, 2, 4^2; n) = 2 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - (1 + (-1)^n) \sum_{d|n} d \left(\frac{8}{d}\right),$$

(e)

$$R_4(1^2, 2, 4; n) = 4 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - (1 + (-1)^n) \sum_{d|n} d \left(\frac{8}{d}\right),$$

(f)

$$R_4(1^3, 8; n) = \begin{cases} 6 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - 16 \sum_{d|\frac{n}{4}} \frac{n/4}{d} \left(\frac{8}{d}\right) \\ \quad -2 \sum_{d|\frac{n}{4}} d \left(\frac{8}{d}\right), & \textit{if } n \equiv 0 \pmod{2}, \\ 6 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right), & \textit{if } n \equiv 1, 5 \pmod{8}, \\ 4 \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right), & \textit{if } n \equiv 3 \pmod{8}, \\ 0, & \textit{if } n \equiv 7 \pmod{8}. \end{cases}$$

(g)

$$R_4(1, 8^3; n) = \begin{cases} \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) - 2 \sum_{d|n} d \left(\frac{8}{d}\right), & \textit{if } n \equiv 0 \pmod{4}, \\ 0, & \textit{if } n \equiv 2 \pmod{4}, \\ \sum_{d|n} \frac{n}{d} \left(\frac{8}{d}\right) + \sum_{d|n} d \left(\frac{8}{d}\right), & \textit{if } n \equiv 1 \pmod{8}, \\ 0, & \textit{if } n \equiv 3, 5, 7 \pmod{8}. \end{cases}$$

(h)

$$R_4(1,2,8^2;n)$$
$$=\begin{cases}\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-2\sum_{d|\frac{n}{4}}d\left(\frac{8}{d}\right), & \text{if } n\equiv 0 \pmod 2,\\ 2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), & \text{if } n\equiv 1,3 \pmod 8,\\ 0, & \text{if } n\equiv 5,7 \pmod 8.\end{cases}$$

(i)

$$R_4(1,2^2,8;n)$$
$$=\begin{cases}2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-2\sum_{d|\frac{n}{4}}d\left(\frac{8}{d}\right), & \text{if } n\equiv 0 \pmod 2,\\ 2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), & \text{if } n\equiv 1,5 \pmod 8,\\ 4\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), & \text{if } n\equiv 3 \pmod 8,\\ 0, & \text{if } n\equiv 7 \pmod 8.\end{cases}$$

(j)

$$R_4(1^2,4,8;n)$$
$$=\begin{cases}2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-2\sum_{d|\frac{n}{4}}d\left(\frac{8}{d}\right), & \text{if } n\equiv 0 \pmod 2,\\ 4\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), & \text{if } n\equiv 1,5 \pmod 8,\\ 0, & \text{if } n\equiv 3,7 \pmod 8.\end{cases}$$

(k)

$$R_4(1,4^2,8;n)$$
$$=\begin{cases}8\sum_{d|\frac{n}{4}}\frac{n/4}{d}\left(\frac{8}{d}\right)-2\sum_{d|\frac{n}{4}}d\left(\frac{8}{d}\right), & \text{if } n\equiv 0 \pmod 2,\\ 2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), & \text{if } n\equiv 1,5 \pmod 8,\\ 0, & \text{if } n\equiv 3,7 \pmod 8,\end{cases}$$

where $\left(\frac{8}{d}\right)$ *is the Legendre-Jacobi-Kronecker symbol for discriminant* 8*, that is,*

$$\left(\frac{8}{d}\right):=\begin{cases}0, & \text{if } d\equiv 0 \pmod 2,\\ 1, & \text{if } d\equiv 1,7 \pmod 8,\\ -1, & \text{if } d\equiv 3,5 \pmod 8.\end{cases}$$

The basic properties of $\varphi(q)$ and $\psi(q)$ are

Proposition 2.2. *(See Berndt [5, p. 15, 71, 72]) Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have*

(a)

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

(b)

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8),$$

(c)

$$\varphi(q)\psi(q^2) = \psi^2(q),$$

(d)

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

(e)

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2),$$

(f)

$$\varphi(-q)\varphi(q) = \varphi^2(-q^2).$$

In Alaca et al. [6, Lemma 2] we find that

$$\varphi^6(q) - \varphi^2(q)\varphi^4(-q) = 16\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2q^n. \tag{2.1}$$

The following relation involving divisor functions

$$\sigma_k(pn) - \left(p^k + 1\right)\sigma_k(n) + p^k\sigma_k(\frac{n}{p}) = 0 \tag{2.2}$$

for a prime p and $k, n \in \mathbb{N}$ is given in Williams [4, Theorem 3.1(ii)].

Proof of Theorem 1.1. (a) In advance it is obvious that

$$q^7\psi(q^8)\psi^3(q^{16}) = 0 \qquad \text{for} \quad n \not\equiv 7 \pmod 8. \tag{2.3}$$

And from Proposition 2.2 (a) and (b) we know that

$$\begin{aligned}
&q^7\psi(q^8)\psi^3(q^{16}) = q\psi(q^8)\cdot\left(q^2\psi(q^{16})\right)^3\\
&= \frac{1}{4}\left(\varphi(q) - \varphi(-q)\right)\left(\frac{1}{4}\left(\varphi(q^2) - \varphi(-q^2)\right)\right)^3\\
&= \frac{1}{4}\left\{\varphi(q) - \left(2\varphi(q^4) - \varphi(q)\right)\right\}\left[\frac{1}{4}\left\{\varphi(q^2) - \left(2\varphi(q^8) - \varphi(q^2)\right)\right\}\right]^3\\
&= \frac{1}{2}\left(\varphi(q) - \varphi(q^4)\right)\left(\frac{1}{2}\left(\varphi(q^2) - \varphi(q^8)\right)\right)^3\\
&= \frac{1}{16}\left(\varphi(q)\varphi^3(q^2) - 3\varphi(q)\varphi^2(q^2)\varphi(q^8) + 3\varphi(q)\varphi(q^2)\varphi^2(q^8)\right.\\
&\quad - \varphi(q)\varphi^3(q^8) - \varphi(q^4)\varphi^3(q^2) + 3\varphi(q^4)\varphi^2(q^2)\varphi(q^8)\\
&\quad \left. -3\varphi(q^4)\varphi(q^2)\varphi^2(q^8) + \varphi(q^4)\varphi^3(q^8)\right)
\end{aligned}$$

thus by (1.7) we obtain

$$
\begin{aligned}
&q^7\psi(q^8)\psi^3(q^{16})\\
&=\frac{1}{16}\left\{\left(1+\sum_{n=1}^{\infty}R_4(1,2^3;n)q^n\right)-3\left(1+\sum_{n=1}^{\infty}R_4(1,2^2,8;n)q^n\right)\right.\\
&\quad+3\left(1+\sum_{n=1}^{\infty}R_4(1,2,8^2;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(1,8^3;n)q^n\right)\\
&\quad-\left(1+\sum_{n=1}^{\infty}R_4(2^3,4;n)q^n\right)+3\left(1+\sum_{n=1}^{\infty}R_4(2^2,4,8;n)q^n\right)\\
&\quad\left.-3\left(1+\sum_{n=1}^{\infty}R_4(2,4,8^2;n)q^n\right)+\left(1+\sum_{n=1}^{\infty}R_4(4,8^3;n)q^n\right)\right\}\\
&=\frac{1}{16}\sum_{n=1}^{\infty}\left\{R_4(1,2^3;n)-3R_4(1,2^2,8;n)+3R_4(1,2,8^2;n)\right.\\
&\quad-R_4(1,8^3;n)-R_4(1^3,2;\frac{n}{2})+3R_4(1^2,2,4;\frac{n}{2})\\
&\quad\left.-3R_4(1,2,4^2;\frac{n}{2})+R_4(1,2^3;\frac{n}{4})\right\}q^n.
\end{aligned}
\tag{2.4}
$$

Appealing to Proposition 2.1 (a), (c), (d), (e), (g), (h), (i), we can write Eq. (2.4) as

$$
\begin{aligned}
&q^7\psi(q^8)\psi^3(q^{16})\\
&=\begin{cases}0, & \text{if } n\equiv 0 \pmod 2,\\ \frac{1}{16}\sum_{n=1}^{\infty}\Big(R_4(1,2^3;n)-3R_4(1,2^2,8;n) & \\ \quad +3R_4(1,2,8^2;n)-R_4(1,8^3;n)\Big)q^n, & \text{if } n\equiv 1 \pmod 2\end{cases}\\
&=\begin{cases}0, & \text{if } n\equiv 0 \pmod 2,\\ \frac{3}{16}\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-\sum_{d|n}d\left(\frac{8}{d}\right)\right)q^n, & \text{if } n\equiv 1 \pmod 8\\ -\frac{1}{8}\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)+\sum_{d|n}d\left(\frac{8}{d}\right)\right)q^n, & \text{if } n\equiv 3,5 \pmod 8)\\ \frac{1}{8}\sum_{n=1}^{\infty}\left(2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-\sum_{d|n}d\left(\frac{8}{d}\right)\right)q^n, & \text{if } n\equiv 7 \pmod 8\end{cases}
\end{aligned}
$$

so combining the above result with (2.3) we have

$$
\begin{aligned}
\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-\sum_{d|n}d\left(\frac{8}{d}\right)=0 \qquad &\text{if } n\equiv 1 \pmod 8,\\
\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)+\sum_{d|n}d\left(\frac{8}{d}\right)=0 \qquad &\text{if } n\equiv 3,5 \pmod 8,
\end{aligned}
$$

and

$$q^7\psi(q^8)\psi^3(q^{16}) = \frac{1}{8}\sum_{\substack{n=1\\ n\equiv 7 \pmod 8}}^{\infty}\left(2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right) - \sum_{d|n} d\left(\frac{8}{d}\right)\right)q^n.$$

This shows that

$$\begin{aligned} q^7\psi(q^8)\psi^3(q^{16}) &= \frac{1}{8}q^7\sum_{\substack{n=7\\ n\equiv 7 \pmod 8}}^{\infty}\left(2\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right) - \sum_{d|n} d\left(\frac{8}{d}\right)\right)q^{n-7} \\ &= \frac{1}{8}q^7\sum_{\substack{k=0\\ k\equiv 0 \pmod 8}}^{\infty}\left(2\sum_{d|(k+7)}\frac{k+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(k+7)} d\left(\frac{8}{d}\right)\right)q^k \\ &= \frac{1}{8}q^7\sum_{n=0}^{\infty}\left(2\sum_{d|(8n+7)}\frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right)q^{8n} \end{aligned}$$

and so

$$\psi(q^8)\psi^3(q^{16}) = \frac{1}{8}\sum_{n=0}^{\infty}\left(2\sum_{d|(8n+7)}\frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right)q^{8n}$$

finally, we employ $q \to q^{\frac{1}{8}}$ to complete the proof.

(b) By Proposition 2.2 (a), (b), (c), and (2.2) we see that

$$\begin{aligned} &q^3\psi^2(q^4)\psi^2(q^8) = \psi^2(q^4)\cdot q^3\psi^2(q^8) \\ &= \varphi(q^4)\psi(q^8)\cdot q^3\psi^2(q^8) \\ &= \varphi(q^4)\left(q\psi(q^8)\right)^3 \\ &= \varphi(q^4)\left(\frac{1}{4}\left(\varphi(q) - \varphi(-q)\right)\right)^3 \\ &= \varphi(q^4)\left[\frac{1}{4}\left\{\varphi(q) - \left(2\varphi(q^4) - \varphi(q)\right)\right\}\right]^3 \\ &= \varphi(q^4)\left\{\frac{1}{2}\left(\varphi(q) - \varphi(q^4)\right)\right\}^3 \\ &= \frac{1}{8}\left(\varphi(q^4)\varphi^3(q) - 3\varphi^2(q)\varphi^2(q^4) + 3\varphi(q)\varphi^3(q^4) - \varphi^4(q^4)\right). \end{aligned} \tag{2.5}$$

Then from Proposition 1.1 and (1.7) we can write Eq. (2.5) as

$$
\begin{aligned}
& q^3\psi^2(q^4)\psi^2(q^8) \\
&= \frac{1}{8}\left\{\left(1+\sum_{n=1}^{\infty} R_4(1^3,4;n)q^n\right) - 3\left(1+\sum_{n=1}^{\infty} R_4(1^2,4^2;n)q^n\right)\right. \\
&\qquad \left. +3\left(1+\sum_{n=1}^{\infty} R_4(1,4^3;n)q^n\right) - \left(1+\sum_{n=1}^{\infty} R_4(4^4;n)q^n\right)\right\} \\
&= \frac{1}{8}\sum_{n=1}^{\infty}\Big(R_4(1^3,4;n) - 3R_4(1^2,4^2;n) + 3R_4(1,4^3;n) - R_4(1^4;\frac{n}{4})\Big)\, q^n \\
&= \frac{1}{8}\sum_{n=1}^{\infty}\left(\left(1-\left(\frac{-4}{n}\right)\right)\sigma_1(n) - 3\sigma_1(\frac{n}{2}) + 2\sigma_1(\frac{n}{4})\right) q^n \\
&= \begin{cases} \frac{1}{8}\sum_{n=1}^{\infty}\left(1-\left(\frac{-4}{n}\right)\right)\sigma_1(n)q^n, & \text{if } n\equiv 1 \pmod 2, \\ \frac{1}{8}\sum_{n=1}^{\infty}\Big(\sigma_1(n) - 3\sigma_1(\frac{n}{2}) + 2\sigma_1(\frac{n}{4})\Big)\, q^n, & \text{if } n\equiv 0 \pmod 2 \end{cases} \\
&= \begin{cases} \frac{1}{4}\sum_{n=1}^{\infty}\sigma_1(n)q^n, & \text{if } n\equiv 3 \pmod 4, \\ 0, & \text{if } n\not\equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} \frac{1}{4}q^3\sum_{n=3}^{\infty}\sigma_1(n)q^{n-3}, & \text{if } n\equiv 3 \pmod 4, \\ 0, & \text{if } n\not\equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} \frac{1}{4}q^3\sum_{k=0}^{\infty}\sigma_1(k+3)q^{k}, & \text{if } k\equiv 0 \pmod 4, \\ 0, & \text{if } k\not\equiv 0 \pmod 4. \end{cases}
\end{aligned}
$$

The above identity follows that

$$
\psi^2(q^4)\psi^2(q^8) = \begin{cases} \frac{1}{4}\sum_{k=0}^{\infty}\sigma_1(k+3)q^{k}, & \text{if } k\equiv 0 \pmod 4, \\ 0, & \text{if } k\not\equiv 0 \pmod 4. \end{cases}
$$

Therefore especially if $k \equiv 0 \pmod 4$, that is $k = 4n$ for $n \in \mathbb{N}_0$ then the above result shows that

$$
\psi^2(q^4)\psi^2(q^8) = \frac{1}{4}\sum_{n=0}^{\infty}\sigma_1(4n+3)q^{4n}
$$

and after reducing $q \to q^{\frac{1}{4}}$ we conclude that

$$
\psi^2(q)\psi^2(q^2) = \frac{1}{4}\sum_{n=0}^{\infty}\sigma_1(4n+3)q^{n}.
$$

(c) Proposition 2.2 (a) and (b) enables us to deduce that

$$\begin{aligned}
&q^5\psi^3(q^8)\psi(q^{16}) = \big(q\psi(q^8)\big)^3 \cdot q^2\psi(q^{16})\\
&= \left(\frac{1}{4}\left(\varphi(q)-\varphi(-q)\right)\right)^3 \cdot \frac{1}{4}\left(\varphi(q^2)-\varphi(-q^2)\right)\\
&= \left[\frac{1}{4}\left\{\varphi(q)-\big(2\varphi(q^4)-\varphi(q)\big)\right\}\right]^3 \cdot \frac{1}{4}\left\{\varphi(q^2)-\big(2\varphi(q^8)-\varphi(q^2)\big)\right\}\\
&= \left(\frac{1}{2}\left(\varphi(q)-\varphi(q^4)\right)\right)^3 \cdot \frac{1}{2}\left(\varphi(q^2)-\varphi(q^8)\right)\\
&= \frac{1}{16}\big(\varphi^3(q)\varphi(q^2)-\varphi^3(q)\varphi(q^8)-3\varphi^2(q)\varphi(q^4)\varphi(q^2)\\
&\quad +3\varphi^2(q)\varphi(q^4)\varphi(q^8)+3\varphi(q)\varphi^2(q^4)\varphi(q^2)-3\varphi(q)\varphi^2(q^4)\varphi(q^8)\\
&\quad -\varphi^3(q^4)\varphi(q^2)+\varphi^3(q^4)\varphi(q^8)\big)
\end{aligned}$$

thus by (1.7), Proposition 2.1 (a), (d), (e), (f), (j), and (k), we obtain

$$\begin{aligned}
&q^5\psi^3(q^8)\psi(q^{16})\\
&= \frac{1}{16}\left\{\left(1+\sum_{n=1}^{\infty}R_4(1^3,2;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(1^3,8;n)q^n\right)\right.\\
&\qquad -3\left(1+\sum_{n=1}^{\infty}R_4(1^2,2,4;n)q^n\right)+3\left(1+\sum_{n=1}^{\infty}R_4(1^2,4,8;n)q^n\right)\\
&\qquad +3\left(1+\sum_{n=1}^{\infty}R_4(1,2,4^2;n)q^n\right)-3\left(1+\sum_{n=1}^{\infty}R_4(1,4^2,8;n)q^n\right)\\
&\qquad \left.-\left(1+\sum_{n=1}^{\infty}R_4(2,4^3;n)q^n\right)+\left(1+\sum_{n=1}^{\infty}R_4(4^3,8;n)q^n\right)\right\}\\
&= \frac{1}{16}\sum_{n=1}^{\infty}\Big\{R_4(1^3,2;n)-R_4(1^3,8;n)-3R_4(1^2,2,4;n)\\
&\qquad +3R_4(1^2,4,8;n)+3R_4(1,2,4^2;n)-3R_4(1,4^2,8;n)\\
&\qquad -R_4(1,2^3;\tfrac{n}{2})+R_4(1^3,2;\tfrac{n}{4})\Big\}q^n\\
&= \begin{cases}
\frac{1}{16}\sum_{n=1}^{\infty}\Big\{R_4(1^3,2;n)-R_4(1^3,8;n) & \\
\quad -3R_4(1^2,2,4;n)+3R_4(1^2,4,8;n) & \\
\quad +3R_4(1,2,4^2;n)-3R_4(1,4^2,8;n)\Big\}q^n, & \text{if } n\equiv 5 \pmod 8,\\
0, & \text{if } n\not\equiv 5 \pmod 8
\end{cases}\\
&= \frac{1}{8}\sum_{\substack{n=1\\ n\equiv 5 \pmod 8}}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)-\sum_{d|n}d\left(\frac{8}{d}\right)\right)q^n.
\end{aligned}$$

The above identity can be written as

$$
\begin{aligned}
&q^5\psi^3(q^8)\psi(q^{16}) \\
&= \frac{1}{8}q^5 \sum_{\substack{n=5\\ n\equiv 5 \pmod 8}}^{\infty} \left(\sum_{d|n} \frac{n}{d}\left(\frac{8}{d}\right) - \sum_{d|n} d\left(\frac{8}{d}\right) \right) q^{n-5} \\
&= \frac{1}{8}q^5 \sum_{\substack{k=0\\ k\equiv 0 \pmod 8}}^{\infty} \left(\sum_{d|(k+5)} \frac{k+5}{d}\left(\frac{8}{d}\right) - \sum_{d|(k+5)} d\left(\frac{8}{d}\right) \right) q^{k} \\
&= \frac{1}{8}q^5 \sum_{n=0}^{\infty} \left(\sum_{d|(8n+5)} \frac{8n+5}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+5)} d\left(\frac{8}{d}\right) \right) q^{8n}
\end{aligned}
$$

and so

$$
\psi^3(q^8)\psi(q^{16}) = \frac{1}{8}\sum_{n=0}^{\infty} \left(\sum_{d|(8n+5)} \frac{8n+5}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+5)} d\left(\frac{8}{d}\right) \right) q^{8n},
$$

which needs $q \to q^{\frac{1}{8}}$ to complete the proof.

□

Proposition 2.3. *(See Alaca et al. [2]) Let $n \in \mathbb{N}$. Then we have*

(a)

$$
R_4(1,4,16^2;n) = \begin{cases} 2\left(1+\left(\frac{-4}{n/4}\right)\right)\sigma_1(\frac{n}{4}) - 2\sigma_1(\frac{n}{8}) \\ \quad +8\sigma_1(\frac{n}{32}) - 32\sigma_1(\frac{n}{64}), & \text{if } n \equiv 0 \pmod 2, \\ \frac{1}{4}\left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n) + \frac{1}{2}\left(2+\left(\frac{2}{n}\right)\right)K_2(n), & \text{if } n \equiv 1 \pmod 2. \end{cases}
$$

(b)

$$
R_4(1,4^2,16;n) = \begin{cases} 2\left(2+\left(\frac{-4}{n/4}\right)\right)\sigma_1(\frac{n}{4}) - 20\sigma_1(\frac{n}{16}) \\ \quad +24\sigma_1(\frac{n}{32}) - 32\sigma_1(\frac{n}{64}), & \text{if } n \equiv 0 \pmod 2, \\ \frac{1}{2}\left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n) + K_2(n), & \text{if } n \equiv 1 \pmod 2. \end{cases}
$$

(c)

$$R_4(1,16^3;n) = \begin{cases} \left(1+\left(\frac{-4}{n/4}\right)\right)\sigma_1(\frac{n}{4}) - 3\sigma_1(\frac{n}{8}) \\ \quad +10\sigma_1(\frac{n}{16}) - 32\sigma_1(\frac{n}{64}), & \text{if } n \equiv 0 \pmod 2, \\ \frac{1}{2}\sigma_1(n) + \frac{3}{2}K_2(n), & \text{if } n \equiv 1 \pmod 8) \\ 0, & \text{if } n \equiv 3,5,7 \pmod 8. \end{cases}$$

We note that $\left(\frac{2}{n}\right)$ *is the Jacobi symbol.*

In the above proposition we use the following method to evaluate $R_4(1, 4^2, 16; n)$ for even n :

$$\begin{aligned} \sum_{\substack{n_1,\cdots,n_4=-\infty \\ 2|n_1}}^{\infty} q^{n_1^2+4n_2^2+4n_3^2+16n_4^2} &= \sum_{n_1,\cdots,n_4=-\infty}^{\infty} q^{4n_1^2+4n_2^2+4n_3^2+16n_4^2} \\ &= 1 + \sum_{n=1}^{\infty} R_4(4^3,16;n)q^n \\ &= 1 + \sum_{n=1}^{\infty} R_4(1^3,4;\frac{n}{4})q^n \end{aligned}$$

and so we refer to Proposition 1.1 (b). In a similar manner we obtain the other formulae for even n. Now we introduce briefly $R_4(a_1, a_2, a_3, a_4; n)$ for $a_1, \cdots a_4 \in \mathbb{N}$ and an odd positive integer n in Proposition 2.4 :

Proposition 2.4. *(See Alaca et al. [2]) Let* $n \in \mathbb{N}$ *be an odd. Then we have*

(a)

$$R_4(1^2,2,8;n) = 2\sigma_1(n) + 2\left(\frac{2}{n}\right)K_2(n),$$

(b)

$$R_4(1^2,8^2;n) = \left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n) + 2\left(\frac{2}{n}\right)K_2(n),$$

(c)

$$R_4(1^2,4,16;n) = \left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n) + 2K_2(n),$$

(d)

$$R_4(1^3,16;n) = \begin{cases} 3\sigma_1(n) + 3K_2(n), & \text{if } n \equiv 1,5 \pmod 8, \\ 2\sigma_1(n), & \text{if } n \equiv 3 \pmod 8) \\ 0, & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Proof of Theorem 1.2. (a) By Proposition 2.2 (a) and (b) we obtain

$$\begin{aligned}
&q^{13}\psi(q^8)\psi^3(q^{32}) = q\psi(q^8)\cdot\left(q^4\psi(q^{32})\right)^3\\
&= \frac{1}{4}\left(\varphi(q)-\varphi(-q)\right)\left(\frac{1}{4}\left(\varphi(q^4)-\varphi(-q^4)\right)\right)^3\\
&= \frac{1}{4}\left\{\varphi(q)-\left(2\varphi(q^4)-\varphi(q)\right)\right\}\left\{\frac{1}{4}\left(\varphi(q^4)-\left(2\varphi(q^{16})-\varphi(q^4)\right)\right)\right\}^3\\
&= \frac{1}{2}\left(\varphi(q)-\varphi(q^4)\right)\left(\frac{1}{2}\left(\varphi(q^4)-\varphi(q^{16})\right)\right)^3\\
&= \frac{1}{16}\big(\varphi(q)\varphi^3(q^4)-3\varphi(q)\varphi^2(q^4)\varphi(q^{16})+3\varphi(q)\varphi(q^4)\varphi^2(q^{16})\\
&\quad -\varphi(q)\varphi^3(q^{16})-\varphi^4(q^4)+3\varphi^3(q^4)\varphi(q^{16})\\
&\quad -3\varphi^2(q^4)\varphi^2(q^{16})+\varphi(q^4)\varphi^3(q^{16})\big)
\end{aligned}$$

so that by Proposition 1.1 (d), (1.7), Proposition 2.3 (a), (b), and (c) we lead that

$$\begin{aligned}
&q^{13}\psi(q^8)\psi^3(q^{32})\\
&= \frac{1}{16}\Bigg\{\left(1+\sum_{n=1}^{\infty}R_4(1,4^3;n)q^n\right)-3\left(1+\sum_{n=1}^{\infty}R_4(1,4^2,16;n)q^n\right)\\
&\qquad +3\left(1+\sum_{n=1}^{\infty}R_4(1,4,16^2;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(1,16^3;n)q^n\right)\\
&\qquad -\left(1+\sum_{n=1}^{\infty}R_4(4^4;n)q^n\right)+3\left(1+\sum_{n=1}^{\infty}R_4(4^3,16;n)q^n\right)\\
&\qquad -3\left(1+\sum_{n=1}^{\infty}R_4(4^2,16^2;n)q^n\right)+\left(1+\sum_{n=1}^{\infty}R_4(4,16^3;n)q^n\right)\Bigg\}\\
&= \frac{1}{16}\sum_{n=1}^{\infty}\Big\{R_4(1,4^3;n)-3R_4(1,4^2,16;n)+3R_4(1,4,16^2;n)\\
&\qquad -R_4(1,16^3;n)-R_4(1^4;\tfrac{n}{4})+3R_4(1^3,4;\tfrac{n}{4})-3R_4(1^2,4^2;\tfrac{n}{4})\\
&\qquad +R_4(1,4^3;\tfrac{n}{4})\Big\}q^n\\
&= \begin{cases}\dfrac{1}{16}\displaystyle\sum_{n=1}^{\infty}\Big\{R_4(1,4^3;n)-3R_4(1,4^2,16;n) & \\ \qquad +3R_4(1,4,16^2;n)-3R_4(1,16^3;n)\Big\}q^n, & \text{if } n\equiv 5 \pmod 8,\\ 0, & \text{if } n\not\equiv 5 \pmod 8\end{cases}\\
&= \frac{1}{64}\sum_{\substack{n=1\\ n\equiv 5 \pmod 8}}^{\infty}\left(\left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n)+6\left(\frac{2}{n}\right)K_2(n)\right)q^n.
\end{aligned}$$

The above equation can be written as

$$\begin{aligned}
&q^{13}\psi(q^8)\psi^3(q^{32})\\
&= \frac{1}{64}q^{13}\sum_{\substack{n=13\\ n\equiv 5 \pmod 8}}^{\infty}\left(\left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n)+6\left(\frac{2}{n}\right)K_2(n)\right)q^{n-13}\\
&= \frac{1}{64}q^{13}\sum_{\substack{k=0\\ k\equiv 0 \pmod 8}}^{\infty}\left(\left(1+\left(\frac{-4}{k+13}\right)\right)\sigma_1(k+13)\right.\\
&\qquad\qquad \left.+6\left(\frac{2}{k+13}\right)K_2(k+13)\right)q^{k} \qquad (2.6)\\
&= \frac{1}{64}q^{13}\sum_{n=0}^{\infty}\left(\left(1+\left(\frac{-4}{8n+13}\right)\right)\sigma_1(8n+13)\right.\\
&\qquad\qquad \left.+6\left(\frac{2}{8n+13}\right)K_2(8n+13)\right)q^{8n}\\
&= \frac{1}{64}q^{13}\sum_{n=0}^{\infty}\left(2\sigma_1(8n+13)-6K_2(8n+13)\right)q^{8n}
\end{aligned}$$

since

$$\left(\frac{-4}{8n+13}\right)=\left(\frac{-4}{4\cdot 2n+4\cdot 3+1}\right)=\left(\frac{-4}{1}\right)=1$$

and

$$\left(\frac{2}{8n+13}\right)=(-1)^{\frac{(8n+13)^2-1}{8}}=(-1)^{8n^2+26n+21}=-1.$$

Therefore Eq. (2.6) shows that

$$\psi(q^8)\psi^3(q^{32})=\frac{1}{64}\sum_{n=0}^{\infty}\left(2\sigma_1(8n+13)-6K_2(8n+13)\right)q^{8n},$$

which requests $q\to q^{\frac{1}{8}}$ to complete the proof.

(b) From Proposition 2.2 (a), (b), (c), and (d) we know that

$$
\begin{aligned}
&q^5\psi^2(q^4)\psi^2(q^{16}) = q\cdot\psi^2(q^4)\cdot q^4\psi^2(q^{16})\\
&= q\cdot\varphi(q^4)\psi(q^8)\cdot\left(q^2\psi(q^{16})\right)^2\\
&= \varphi(q^4)\cdot q\psi(q^8)\cdot\left(q^2\psi(q^{16})\right)^2\\
&= \frac{1}{2}\left(\varphi(q)+\varphi(-q)\right)\cdot\frac{1}{4}\left(\varphi(q)-\varphi(-q)\right)\left(\frac{1}{4}\left(\varphi(q^2)-\varphi(-q^2)\right)\right)^2\\
&= \frac{1}{8}\left(\varphi^2(q)-\varphi^2(-q)\right)\left(\frac{1}{4}\left(\varphi(q^2)-\varphi(-q^2)\right)\right)^2\\
&= \frac{1}{8}\left\{\varphi^2(q)-\left(2\varphi^2(q^2)-\varphi^2(q)\right)\right\}\left\{\frac{1}{4}\left(\varphi(q^2)-\left(2\varphi(q^8)-\varphi(q^2)\right)\right)\right\}^2\\
&= \frac{1}{4}\left(\varphi^2(q)-\varphi^2(q^2)\right)\left(\frac{1}{2}\left(\varphi(q^2)-\varphi(q^8)\right)\right)^2\\
&= \frac{1}{16}\left(\varphi^2(q)\varphi^2(q^2)-2\varphi^2(q)\varphi(q^2)\varphi(q^8)+\varphi^2(q)\varphi^2(q^8)-\varphi^4(q^2)\right.\\
&\qquad\left.+2\varphi^3(q^2)\varphi(q^8)-\varphi^2(q^2)\varphi^2(q^8)\right).
\end{aligned}
$$

Therefore by (1.7), Proposition 2.1 (b), Proposition 2.4 (a), and (b) the above identity can be written as

$$
\begin{aligned}
&q^5\psi^2(q^4)\psi^2(q^{16})\\
&= \frac{1}{16}\left\{\left(1+\sum_{n=1}^{\infty}R_4(1^2,2^2;n)q^n\right)-2\left(1+\sum_{n=1}^{\infty}R_4(1^2,2,8;n)q^n\right)\right.\\
&\qquad+\left(1+\sum_{n=1}^{\infty}R_4(1^2,8^2;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(2^4;n)q^n\right)\\
&\qquad\left.+2\left(1+\sum_{n=1}^{\infty}R_4(2^3,8;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(2^2,8^2;n)q^n\right)\right\}\\
&= \frac{1}{16}\sum_{n=1}^{\infty}\left\{R_4(1^2,2^2;n)-2R_4(1^2,2,8;n)+R_4(1^2,8^2;n)\right.\\
&\qquad\left.-R_4(1^4;\frac{n}{2})+2R_4(1^3,4;\frac{n}{2})-R_4(1^2,4^2;\frac{n}{2})\right\}q^n\\
&= \begin{cases}\dfrac{1}{16}\displaystyle\sum_{n=1}^{\infty}\left(R_4(1^2,2^2;n)-2R_4(1^2,2,8;n)\right.\\ \qquad\left.+R_4(1^2,8^2;n)\right)q^n, & \text{if } n\equiv 1 \pmod 4,\\ \\ 0, & \text{if } n\not\equiv 1 \pmod 4\end{cases}\\
&= \frac{1}{16}\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty}\left\{\left(1+\left(\frac{-4}{n}\right)\right)\sigma_1(n)-2\left(\frac{2}{n}\right)K_2(n)\right\}q^n
\end{aligned}
$$

and so

$$\begin{aligned}
&q^5\psi^2(q^4)\psi^2(q^{16})\\
&= \frac{1}{8}\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty}\left(\sigma_1(n)-\left(\frac{2}{n}\right)K_2(n)\right)q^n\\
&= \frac{1}{8}\sum_{k=0}^{\infty}\left(\sigma_1(4k+1)-\left(\frac{2}{4k+1}\right)K_2(4k+1)\right)q^{4k+1}\\
&= q^5\cdot\frac{1}{8}\sum_{k=1}^{\infty}\left(\sigma_1(4k+1)-\left(\frac{2}{4k+1}\right)K_2(4k+1)\right)q^{4(k-1)},
\end{aligned}$$

which shows that

$$\begin{aligned}
&\psi^2(q^4)\psi^2(q^{16})\\
&= \frac{1}{8}\sum_{k=1}^{\infty}\left(\sigma_1(4k+1)-\left(\frac{2}{4k+1}\right)K_2(4k+1)\right)q^{4(k-1)}\\
&= \frac{1}{8}\sum_{n=0}^{\infty}\left(\sigma_1(4n+5)-\left(\frac{2}{4n+5}\right)K_2(4n+5)\right)q^{4n}\\
&= \frac{1}{8}\sum_{n=0}^{\infty}\left(\sigma_1(4n+5)+(-1)^nK_2(4n+5)\right)q^{4n}
\end{aligned} \tag{2.7}$$

since

$$\left(\frac{2}{4n+5}\right)=(-1)^{\frac{(4n+5)^2-1}{8}}=(-1)^{2n^2+5n+3}=(-1)^{n+1}.$$

Finally we apply $q\to q^{\frac{1}{4}}$ to Eq. (2.7).

(c) By Proposition 2.2 (a) and (b) we observe that

$$\begin{aligned}
&q^7\psi^3(q^8)\psi(q^{32})=\left(q\psi(q^8)\right)^3\cdot q^4\psi(q^{32})\\
&=\left(\frac{1}{4}\left(\varphi(q)-\varphi(-q)\right)\right)^3\cdot\frac{1}{4}\left(\varphi(q^4)-\varphi(-q^4)\right)\\
&=\left[\frac{1}{4}\left\{\varphi(q)-\left(2\varphi(q^4)-\varphi(q)\right)\right\}\right]^3\cdot\frac{1}{4}\left\{\varphi(q^4)-\left(2\varphi(q^{16})-\varphi(q^4)\right)\right\}\\
&=\left(\frac{1}{2}\left(\varphi(q)-\varphi(q^4)\right)\right)^3\cdot\frac{1}{2}\left(\varphi(q^4)-\varphi(q^{16})\right)\\
&=\frac{1}{16}\left(\varphi^3(q)\varphi(q^4)-\varphi^3(q)\varphi(q^{16})-3\varphi^2(q)\varphi^2(q^4)+3\varphi^2(q)\varphi(q^4)\varphi(q^{16})\right.\\
&\quad\left.+3\varphi(q)\varphi^3(q^4)-3\varphi(q)\varphi^2(q^4)\varphi(q^{16})-\varphi^4(q^4)+\varphi^3(q^4)\varphi(q^{16})\right).
\end{aligned}$$

Then by Proposition 1.1 (b), (c), (d), (1.7), Proposition 2.3 (b), Proposition 2.4 (c), and (d) we can rewrite the above identity as

$$\begin{aligned}
&q^7\psi^3(q^8)\psi(q^{32})\\
&=\frac{1}{16}\left\{\left(1+\sum_{n=1}^{\infty}R_4(1^3,4;n)q^n\right)-\left(1+\sum_{n=1}^{\infty}R_4(1^3,16;n)q^n\right)\right.\\
&\quad-3\left(1+\sum_{n=1}^{\infty}R_4(1^2,4^2;n)q^n\right)+3\left(1+\sum_{n=1}^{\infty}R_4(1^2,4,16;n)q^n\right)\\
&\quad+3\left(1+\sum_{n=1}^{\infty}R_4(1,4^3;n)q^n\right)-3\left(1+\sum_{n=1}^{\infty}R_4(1,4^2,16;n)q^n\right)\\
&\quad\left.-\left(1+\sum_{n=1}^{\infty}R_4(4^4;n)q^n\right)+\left(1+\sum_{n=1}^{\infty}R_4(4^3,16;n)q^n\right)\right\}\\
&=\frac{1}{16}\sum_{n=1}^{\infty}\Big\{R_4(1^3,4;n)-R_4(1^3,16;n)-3R_4(1^2,4^2;n)\\
&\qquad+3R_4(1^2,4,16;n)+3R_4(1,4^3;n)-3R_4(1,4^2,16;n)\\
&\qquad-R_4(1^4;\frac{n}{4})+R_4(1^3,4;\frac{n}{4})\Big\}q^n\\
&=\begin{cases}\frac{1}{16}\sum_{n=1}^{\infty}\big(R_4(1^3,4;n)-R_4(1^3,16;n)\\ \quad-3R_4(1^2,4^2;n)+3R_4(1^2,4,16;n)\\ \quad+3R_4(1,4^3;n)-3R_4(1,4^2,16;n)\big)\,q^n, & \text{if } n\equiv 7 \pmod 8,\\ 0, & \text{if } n\not\equiv 7 \pmod 8\end{cases}\\
&=\frac{1}{32}\sum_{\substack{n=1\\ n\equiv 7 \ (\mathrm{mod}\ 8)}}^{\infty}\left\{\left(5+\left(\frac{-4}{n}\right)\right)\sigma_1(n)+6K_2(n)\right\}q^n.
\end{aligned} \tag{2.8}$$

Since for $n \equiv 7 \pmod 8$ and (1.4) we have

$$\left(\frac{-4}{n}\right)=\left(\frac{-4}{3}\right)=-1 \qquad \text{and} \qquad K_2(n)=0$$

thus Eq. (2.8) becomes

$$\begin{aligned}
q^7\psi^3(q^8)\psi(q^{32}) &= \frac{1}{8}\sum_{\substack{n=1\\ n\equiv 7 \ (\mathrm{mod}\ 8)}}^{\infty}\sigma_1(n)q^n\\
&=\frac{1}{8}q^7\sum_{\substack{n=7\\ n\equiv 7 \ (\mathrm{mod}\ 8)}}^{\infty}\sigma_1(n)q^{n-7}\\
&=\frac{1}{8}q^7\sum_{\substack{k=0\\ k\equiv 0 \ (\mathrm{mod}\ 8)}}^{\infty}\sigma_1(k+7)q^k.
\end{aligned}$$

Finally the above equation shows that

$$\psi^3(q^8)\psi(q^{32}) = \frac{1}{8}\sum_{\substack{k=0\\ k\equiv 0 \pmod 8}}^{\infty} \sigma_1(k+7)q^k$$
$$= \frac{1}{8}\sum_{n=0}^{\infty}\sigma_1(8n+7)q^{8n}$$

and so we apply $q \to q^{\frac{1}{8}}$.

□

Proof of Theorem 1.3. (a) From (1.2), (1.6), (1.8), and Theorem 1.1 (a) we easily know that

$$\sum_{n=0}^{\infty} N_4(\frac{1}{2}, 1^3; n)q^n$$
$$= \sum_{n_1,\cdots,n_4=0}^{\infty} q^{\frac{1}{2}n_1(n_1+1)+n_2(n_2+1)+n_3(n_3+1)+n_4(n_4+1)}$$
$$= \psi(q)\psi^3(q^2)$$
$$= \frac{1}{8}\sum_{n=0}^{\infty}\left(2\sum_{d|(8n+7)}\frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right)q^n.$$

(b) Now we have in Theorem 1.1 (b)

$$\psi^2(q)\psi^2(q^2) = \frac{1}{4}\sum_{n=1}^{\infty}\sigma_1(4n+3)q^n,$$

which shows by (1.6) and (1.8)

$$\sum_{n=0}^{\infty} N_4(\frac{1}{2}, \frac{1}{2}, 1, 1; n)q^n$$
$$= \sum_{n_1,\cdots,n_4=0}^{\infty} q^{\frac{1}{2}n_1(n_1+1)+\frac{1}{2}n_2(n_2+1)+n_3(n_3+1)+n_4(n_4+1)}$$
$$= \psi^2(q)\psi^2(q^2)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty}\sigma_1(4n+3)q^n.$$

(c) It is definite by Theorem 1.1 (c).

(d) It is clear by Theorem 1.2 (a).

(e) It is obvious by Theorem 1.2 (b).

(f) It is definite by Theorem 1.2 (c).

□

3 Conclusion

In this article we aim to deduce the number of representations of a positive integer as a sum of four triangular numbers for example,

$$N_4(\frac{1}{2}, 1^3; n) = \frac{1}{8}\left(2\sum_{d|(8n+7)} \frac{8n+7}{d}\left(\frac{8}{d}\right) - \sum_{d|(8n+7)} d\left(\frac{8}{d}\right)\right),$$

$$N_4(\left(\frac{1}{2}\right)^2, 1^2; n) = \frac{1}{4}\sigma_1(4n+3),$$

and etc.

Competing Interests

Author has declared that no competing interests exist.

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APPENDIX

The first twenty values of $K_2(n)$ are listed in the following table.

Table 1. $K_2(n)$ **for** n $(1 \leq n \leq 20)$

n	$K_2(n)$	n	$K_2(n)$	n	$K_2(n)$	n	$K_2(n)$
1	1	6	0	11	0	16	0
2	0	7	0	12	0	17	2
3	0	8	0	13	-6	18	0
4	0	9	-3	14	0	19	0
5	2	10	0	15	0	20	0

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