British Journal of Mathematics & Computer Science
19(4): 1-13, 2016; Article no.BJMCS.29786
ISSN: 2231-0851

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On Some Properties of Solutions of the p-Harmonic Equation in Unbounded Domains

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/29786
Editor(s):
(1) Andrej V. Plotnikov, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
Reviewers:
(1) Abdullah Akkurt, University of Kahramanmaras Sutcu Imam, Turkey.
(2) Shaolin Chen, Hengyang Normal University, China.
(3) Chengyuan Qu, Dalian Minzu University, China.
Complete Peer review History: http://www.sciencedomain.org/review-history/16797

Received: 29th September 2016
Accepted: 26th October 2016
Published: 4th November 2016

Original Research Article

Abstract

We shall formulate some properties, as Phragmén-Lindelöf theorem and asymptotic behavior at infinity, for solutions of the p-Laplacean equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\right) = f(x) \quad (p > 0),$$

in an unbounded domain Q of $\mathbb{R}^n$ $(n \geq 2)$.

Keywords: p-Laplacean; Phragmén-Lindelöf theorem; asymptotic behavior of solutions; unbounded domain.

2010 Mathematics Subject Classification: 35J92, 35B40, 35B53.

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1 Introduction

We consider the solutions to the p-Laplacean equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\right) = f(x) \quad (p>0), \tag{1.1}$$

in $Q \subset \mathbb{R}^n$.

The existence and uniqueness of solution for boundary value problem related to equation (1.1) have been obtained by many authors, see for instance [1], and [2].

We study some properties of solutions of (1.1) at infinity supposing that Q is a cylindrical or conical or more general unbounded domain of $\mathbb{R}^n$ $(n \geq 2)$.

In particular, we shall show that a theorem of kind Phragmén-Lindelöf it holds for solutions of equation (1.1) in cylindrical domain

$$\pi_0 = \{x = (x', x_n) \in \mathbb{R}^n : \ x' \in \Omega, \ x_n > 0\},$$

where $x' = (x_1, ..., x_{n-1})$ and Ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary $\partial\Omega$. The analogous question, for 2m-order linear equation, was first investigated by P.D. Lax in [3]; more precisely, Lax, considering in π_0 the solution $u(x)$ of an elliptic higher-order equation with constant coefficients and Dirichlet-data zero on

$$\sigma_0 = \{x = (x', x_n) \in \mathbb{R}^n : \ x' \in \partial\Omega, \ x_n > 0\},$$

assuming, moreover, that

$$\int_{\pi_0} \sum_{|\alpha|=m} |D^\alpha u|^2 \, dx < +\infty,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, has proved that there exists a constant $\beta > 0$ such that

$$\int_{\pi_0} e^{\beta x_n} \sum_{|\alpha|=m} |D^\alpha u|^2 \, dx < +\infty.$$

We also treat the Neumann problem and extend such results to the case where Q is a conical unbounded domain of $\mathbb{R}^n$. In [4], S. Agmon and L. Nirenberg have dealt analogous problems for ordinary differential equations in Hilbert spaces.

For other discussions of Phragmén-Lindelöf principles see [5], [6] and the book of Protter and Weinberger [7].

Finally, we shall study the asymptotic behavior of solutions of equation (1.1) in an unbounded domain contained in

$$S_1 = \left\{x = (x', x_n) \in \mathbb{R}^n : 1 < x_n < +\infty, \ |x'|^2 < x_n^m \quad (0 < m < 1)\right\}.$$

Recently, the asymptotic behavior of solutions have been exploited in a significant number of articles (see, for instance, [8], [9], [10] and the references given there).

2 Preliminaries

Let us denote by $\pi_{a,b}$, $\sigma_{a,b}$ $(0 \le a < b \le +\infty)$ the sets

$$\pi_{a,b} = \{x = (x', x_n) \in \mathbb{R}^n : \; x' \in \Omega, \; a < x_n < b\},$$

$$\sigma_{a,b} = \{x = (x', x_n) \in \mathbb{R}^n : \; x' \in \partial\Omega, \; a < x_n < b\},$$

where $x' = (x_1, ..., x_{n-1})$ and Ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary $\partial\Omega$; $\pi_a = \pi_{a,\infty}$, $\sigma_a = \sigma_{a,\infty}$.

We shall suppose that $f(x)$ is bounded function. In the sequel, by c_i $(i = 1, 2, ..., 14)$, γ_j $(j = 1, 2, 3, 4)$ we shall denote positive constants depending only on n, p and known parameters. Moreover, for example, to indicate a dependence of α on the real parameters n, p and meas Ω we shall write $\alpha = \alpha(n, p, \Omega)$.

3 New Results

Theorem (3.1). *Let $u(x)$ be a solution of* (1.1) *in π_0, $u(x) = 0$ on σ_0. Let us suppose that*

$$A = \int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx < +\infty \text{ and,}$$

$f(x) = 0$ in π_a for some $a > 0$. Then there exists a positive constant α_1, $\alpha_1 = \alpha_1(n, p, \Omega)$, such that

$$\int_{\pi_0} \left(e^{\alpha_1 x_n} |u|^{p+1} + e^{\alpha_1 x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} \right) dx < +\infty.$$

Proof.- For any a, b such that $0 \le a < b \le +\infty$ set

$$I_{a,b}(u) = \int_{\pi_{a,b}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx, \; I_a(u) = I_{a,\infty}(u).$$

For the sake of simplicity, we will assume throughout that $f(x) = 0$.

Let $\theta(x_n) \in C^1(\mathbb{R})$ be a function such that $\theta(x_n) = 1$ if $x_n < \frac{1}{2}$, $\theta(x_n) = 0$ if $x_n > 1$, $0 \le \theta(x_n) \le 1$, $|\theta'(x_n)| \le \Gamma$. For every $a > 0$, we consider $\theta_a(x_n) = \theta(x_n - a)$.

Let a be a real non-negative numbers. Let us prove that, for all $b > a$,

$$(3.1) \qquad \int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx = \int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-1} \frac{\partial u}{\partial x_i} \frac{\partial(\theta_b(x_n)u)}{\partial x_i} dx.$$

Really, $(\theta_c(x_n) - \theta_b(x_n))u \in \mathring{W}^{1,p+1}(\pi_{b+\frac{1}{2},c+1})$ if $c > b > a$. According to equation (1.1), this implies

$$\int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-1} \frac{\partial u}{\partial x_i} \frac{\partial(\theta_c(x_n)u - \theta_b(x_n)u)}{\partial x_i} dx = 0;$$

therefore, the right-hand side in (3.1) does not depend on b.

At the same time, we have

$$\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial(\theta_b(x_n)u)}{\partial x_i}dx=\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}\theta_b(x_n)dx+ \tag{3.2}$$
$$+\int_{\pi_0}\sum_{i=1}^{n}u\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial\theta_b(x_n)}{\partial x_i}dx.$$

It is obvious that

$$\lim_{b\to+\infty}\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}\theta_b(x_n)dx=\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx. \tag{3.3}$$

Let us estimate the second summand on the right in (3.2).

By the Hölder inequality, we obtain

$$\left|\int_{\pi_0}\sum_{i=1}^{n}u\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial\theta_b(x_n)}{\partial x_i}dx\right|\le \tag{3.4}$$
$$\le\left(\int_{\pi_{b+\frac{1}{2},b+1}}\left|\frac{\partial u}{\partial x_n}\right|^{p+1}dx\right)^{\frac{p}{p+1}}\left(\int_{\pi_{b+\frac{1}{2},b+1}}|u|^{p+1}|\theta_b'(x_n)|^{p+1}dx\right)^{\frac{1}{p+1}}.$$

According to Friedrichs inequality (see, for instance,[11], [12]), the following estimate is valid:

$$\int_{\Omega}|u(x')|^{p+1}dx'\le c(n,p,\Omega)\int_{\Omega}\sum_{i=1}^{n-1}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx'. \tag{F}$$

On the other hand $|\theta_b'(x_n)|\le\Gamma$ for every $b>0$ and $x_n>0$.

Consequently, we have

$$\left(\int_{\pi_{b+\frac{1}{2},b+1}}|u|^{p+1}|\theta_b'(x_n)|^{p+1}dx\right)\le\Gamma^{p+1}c(n,p,\Omega)\int_{\pi_{b+\frac{1}{2},b+1}}\sum_{i=1}^{n-1}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$\int_{\pi_0}\sum_{i=1}^{n}u\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial\theta_b(x_n)}{\partial x_i}dx\to 0\text{ as }b\to+\infty.$$

Thus, estimate (3.1) is proved.

Further, relations (3.1) and (3.2) imply the formula

$$\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx=\int_{\pi_0}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}\theta_b(x_n)dx+$$
$$+\int_{\pi_0}\sum_{i=1}^{n}u\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial\theta_b(x_n)}{\partial x_i}dx$$

for all $b>a$. At the same time, from (3.4) and (3.5), it follows that

$$\left|\int_{\pi_0}\sum_{i=1}^{n}u\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial u}{\partial x_i}\frac{\partial\theta_b(x_n)}{\partial x_i}dx\right|\le\Gamma[c(n,p,\Omega)]^{\frac{1}{p+1}}\int_{\pi_{b+\frac{1}{2},b+1}}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx.$$

Therefore, for all $b > a$,

$$\int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq \int_{\pi_0} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} \theta_b dx + \tag{3.6}$$

$$+\alpha_0 \int_{\pi_{b+\frac{1}{2},b+1}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx,$$

where the constant $\alpha_0 = \Gamma[c(n,p,\Omega)]^{\frac{1}{p+1}}$ does not depend on u and b.

If $f(x)$ does not equal to 0 in π_0 we know that $f = 0$ in π_a for $a > a^*$. As is shown above, for every $b > a^*$, formula (3.6) is valid. Hence, we have

$$\int_{\pi_{b+\frac{1}{2}}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq (1+\alpha_0) \int_{\pi_{b+\frac{1}{2},b+1}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx$$

for all $b > a^*$.

Last inequality implies

$$I_{b+1}(u) \leq \frac{\alpha_0}{\alpha_0 + 1} I_b(u), \ \forall b > a^*.$$

This formula, by induction, gives

$$I_{b+m}(u) \leq s^m I_b(u) \leq A s^m,$$

for $m \in \mathbb{N}$, $b > a^*$ and $s = \frac{\alpha_0}{\alpha_0+1}$. Now, we can write last relation in this way

$$I_{b+m}(u) \leq A e^{m \log s}, \text{ for any } b > a^*, \ m \in \mathbb{N} \cup \{0\}.$$

It is simple to verify that last inequality gives the following

$$I_\lambda(u) \leq c_3 e^{-\lambda \tilde{\alpha}}, \text{ for all } \lambda > 0,$$

where $c_3 = A e^{(1+a^*)\tilde{\alpha}}$ and $\tilde{\alpha} = -\log s > 0$.

Next, fix α_1: $0 < \alpha_1 < \tilde{\alpha}$. We have:

$$\int_{\pi_0} e^{\alpha_1 x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx = \sum_{j=0}^{+\infty} \int_{\pi_{j,j+1}} e^{\alpha_1 x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq$$

$$\leq \sum_{j=0}^{+\infty} e^{\alpha_1 (j+1)} \int_{\pi_{j,j+1}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq \sum_{j=0}^{+\infty} e^{\alpha_1 (j+1)} I_j(u) \leq$$

$$\leq c_3 \sum_{j=0}^{+\infty} e^{\alpha_1 (j+1)} e^{-j\tilde{\alpha}} < +\infty.$$

Finally, an other application of Friedrichs inequality gives us the required conclusion.

Remark (3.2). From (3.1) it is easy to prove that there exists a constant $\gamma_1 > 1$ such that

$$I_b(u) \leq \gamma_1 I_b(\theta_b(x_n)u)$$

for b sufficiently large.

Neumann problem

Now, we will consider a weak solution $u(x)$ of (1.1) in π_0 with the boundary condition

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^p \cos\Theta_i = 0 \text{ on } \sigma_0, \tag{3.7}$$

Θ_i is the angle between the axis x_i and the direction of the outer normal vector on $\partial\Omega$.

Theorem (3.3). *Let $u(x)$ be a solution of* $(1.1)-(3.7)$. *Let us suppose that*

$$A = \int_{\pi_0} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx < +\infty \textit{ and,}$$

$f(x) = 0$ in π_a for some $a > 0$. Then there exist two constants $\alpha_2 > 0$, $\alpha_2 = \alpha_2(n, p, \Omega)$, and h such that

$$\int_{\pi_0} \left(e^{\alpha_2 x_n}|u(x) - h|^{p+1} + e^{\alpha_2 x_n} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx \right) < +\infty.$$

Proof.- Put $\overline{u} = (\text{meas } \pi_{b,b+1})^{-1} \int_{\pi_{b,b+1}} u dx$. Arguing as Theorem (3.1) (see remark (3.2)) we can prove that there exists a constant $\gamma_2 > 1$ such that

$$I_b(u) \leq \gamma_2 I_b(\theta_b(x_n)(u - \overline{u})) \tag{3.8}$$

for b sufficiently large. From this relation and Poincaré - Wirtinger inequality we obtain

$$I_{b+1}(u) \leq A(1 - c_4^{-1}) \ (c_4 > 1)$$

and, by the same procedure as in the proof of the Theorem (3.1), we prove that there exists a positive constant $\alpha = \alpha(n, p, \Omega)$ such that

$$I(u) = \sum_{i=1}^{n} \int_{\pi_0} e^{\alpha x_n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx < +\infty.$$

Next, we define in $(0, +\infty)$ the function

$$v(x_n) = (\text{meas } \Omega)^{-1} \int_{\Omega} u(x', x_n) dx'.$$

From Hölder-Riesz inequality, it follows

$$\int_0^{+\infty} e^{\alpha x_n} |v'(x_n)|^{p+1} dx_n \leq \frac{1}{\text{meas } \Omega} I(u) < +\infty.$$

Hence, if we change variables $t = e^{x_n}$ we have

$$\int_1^{+\infty} t^{\alpha+p} |\tilde{v}'(t)|^{p+1} dt = \int_0^{+\infty} e^{\alpha x_n} |v'(x_n)|^{p+1} dx_n < +\infty,$$

where $\tilde{v}(t) = v(\log t)$. From the Hardy classical inequality (see, for instance [13]) we can state that there exists a constant h such that

$$\int_1^{+\infty} t^{\alpha-1}|\tilde{v}(t)-h|^{p+1}dt \le \left(\frac{p+1}{\alpha}\right)^{p+1}\int_1^{+\infty}|\tilde{v}'(t)|^{p+1}t^{\alpha+p}dt.$$

A new change of variable gives

$$\int_0^{+\infty} e^{\alpha x_n}|v(x_n)-h|^{p+1}dx_n \le \left(\frac{p+1}{\alpha}\right)^{p+1}\frac{1}{\text{meas }\Omega}\int_{\pi_0} e^{\alpha x_n}\left|\frac{\partial u}{\partial x_n}\right|^{p+1}dx.$$

Integrating the last relation on Ω we obtain

$$\int_{\pi_0} e^{\alpha x_n}|v(x_n)-h|^{p+1}dx \le \left(\frac{p+1}{\alpha}\right)^{p+1} I(u). \tag{3.9}$$

Finally, the Poincaré - Wirtinger inequality implies

$$\int_\Omega |u-v(x_n)|^{p+1}dx' \le c_5\int_\Omega \sum_{i=1}^{n-1}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx'$$

and so,

$$\int_{\pi_0} e^{\alpha x_n}|u-v(x_n)|^{p+1}dx \le c_6 I(u). \tag{3.10}$$

Obviously inequalities (3.9) and (3.10) conclude our Theorem.

Now, we shall consider weak solutions of (1.1) in a conical unbounded domain. Let K a cone of $\mathbb{R}^n$; $\forall\ a,\ b : 0 \le a < b \le +\infty$ we define

$$K_{a,b} = \{x \in \mathbb{R}^n : x \in K, a < |x| < b\},\ K_a = K_{a,+\infty}$$

$$FK_{a,b} = \{x \in \mathbb{R}^n : x \in \partial K, a < |x| < b\},\ FK_a = FK_{a,+\infty}.$$

Theorem (3.4). *Let $u(x)$ be a weak solution of (1.1) in K_1 such that $u(x) = 0$ on FK_1. Let us suppose that*

$$A = \int_{K_1}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx < +\infty \text{ and,}$$

$f(x) = 0$ in K_R for some $R \ge 1$. Then there exist a constant $\alpha_3 > 0$, $\alpha_3 = \alpha_3(n, p, K_{1,2})$, such that

$$\int_{K_1}|x|^{\alpha_3-(p+1)}|u|^{p+1}dx + \int_{K_1}|x|^{\alpha_3}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p+1}dx < +\infty.$$

Proof.- We assume $f = 0$ in K_R, for $R > R^*$. Let $\theta(x) \in C^1(\mathbb{R})$ be a function such that $\theta(x) = 1$ if $x < 1$, $\theta(x) = 0$ if $x > 2$, $0 \le \theta(x_n) \le 1$, $|\theta'(x)| \le \beta$.

For every $R \ge 1$, we consider $\theta_R(x) = \theta_R(|x|) = \theta(\frac{|x|}{R})$. It results $0 \le \theta_R(x) \le 1$ and $|\nabla\theta_R(x)| \le \frac{\beta}{R}\ \forall\ R \ge 1$.

Arguing as in previous theorems, since

$$\int_{K_1}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^{p-1}\frac{\partial\left[u(\theta_{2R}-\theta_R)\right]}{\partial x_i}dx = 0,\ \text{for } R > R^*,$$

we obtain a constant $\gamma_3 > 1$, independent of $u(x)$, such that

$$\int_{K_R} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq \gamma_3 \int_{K_R} \sum_{i=1}^{n} \left| \frac{\partial \theta_R(x) u}{\partial x_i} \right|^{p+1} dx \tag{3.11}$$

for $R > R^*$. From Friedrichs inequality ($u = 0$ on $\partial K \setminus \{x \in \mathbb{R}^n : |x| < 1\}$), applied in the cone $K_{1,2}$ and the change of variables $Rx = x'$, we have

$$\int_{K_{R,2R}} |u|^{p+1} dx \leq c_6 R^{p+1} \int_{K_{R,2R}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx. \tag{3.12}$$

From (3.11) and (3.12) we obtain

$$\int_{K_R} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq c_7 \int_{K_{R,2R}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx,$$

for $R > R^*$ and $c_7 > 1$. It results

$$\int_{K_{2R}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx = \int_{K_R} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx -$$

$$- \int_{K_{R,2R}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq \left(1 - \frac{1}{c_7}\right) \int_{K_R} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

Now, if we put $R = 1, 2, ..., 2^N, ..,$ from last inequality we have

$$\int_{K_{2^N}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq A\rho^N$$

for $N > N^\star$ and $\rho = \left(1 - \frac{1}{c_7}\right) \in]0, 1[$.

Fix $\alpha_3 : 0 < \alpha_3 < -\log_2 \rho$. It results

$$\int_{K_1} |x|^{\alpha_3} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq 2^{\alpha_3 N^\star} A + \sum_{N=N^\star}^{+\infty} \int_{K_{2^N, 2^{N+1}}} |x|^{\alpha_3} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq$$

$$\leq 2^{\alpha_3 N^\star} A + \sum_{N=N^\star}^{+\infty} 2^{\alpha_3(N+1)} \int_{K_{2^N}} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq$$

$$\leq A \left(2^{\alpha_3 N^\star} + \sum_{N=0}^{+\infty} 2^{\alpha_3(N+1)} \rho^N \right) < +\infty.$$

Finally, we conclude our theorem applying the following Hardy weighted inequality (see, for instance, [13])

$$\int_{K_1} |x|^{\alpha_3 - (p+1)} |u|^{p+1} dx \leq c_8 \int_{K_1} |x|^{\alpha_3} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

Remark (3.5). Theorem (3.4) holds for solutions of Neumann problem in the conical domain K_1. The proof is similar to theorem (3.3), it is possible to use Poincaré-Wirtinger and Hardy inequalities instead of Friedrichs inequality.

The constant α_1 of Theorem (3.1) does not depend on $u(x)$ but it depends on Kondratiev - Lax constant $c(n,p,\Omega)$ present in (F), then it depends on meas Ω. It is important to note that $\alpha_1 = \alpha_1(\Omega) \to +\infty$ as meas $\Omega \to 0$. Analogous considerations for the constant α_2 and α_3 of Theorems (3.3) and (3.4) respectively.

Now, we consider solutions to the equation (1.1) in unbounded domain S such that

$$S \subseteq S_1 = \left\{ x = (x', x_n) \in \mathbb{R}^n : 1 < x_n < +\infty,\ |x'|^2 = \sum_{i=1}^{n-1} x_i^2 < x_n^m \ (0 < m < 1) \right\}.$$

Assuming that $u = 0$ on ∂S and $A = \int_S \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx < +\infty$ we shall obtain

Theorem (3.6) (Asymptotic behavior). *There exists a constant $\delta > 0$ independent of S_1 and $u(x)$ such that*

$$\int_{\{x \in S : x_n > 2^t\}} \left(|u|^{p+1} + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} \right) dx \le c_{11} e^{-\delta t^{1-m}}, \ \forall\ t \text{ large enough.}$$

Proof.- We put $u(x) = 0$ in $S_1 \setminus S$ and we introduce a function $\tau(x_n) \in C^1(\mathbb{R})$ such that $\tau(x_n) = 1$ if $x_n < 0$, $\tau(x_n) = 0$ if $x_n > 1$, $0 \le \tau(x_n) \le 1$, $|\tau'(x_n)| \le \beta_1$.

For every $\lambda \ge 1$, we consider $\theta_\lambda(x_n) = \tau(\frac{x_n - \lambda}{\lambda^m})$. It results $0 \le \theta_\lambda(x_n) \le 1$ and $|\theta'_\lambda(x_n)| \le \frac{\beta_1}{\lambda^m}$ $\forall\ \lambda \ge 1$.

Moreover

$$\theta_\lambda(x_n) = \begin{cases} 0 & if \quad x_n > \lambda + \lambda^m \\ 1 & if \quad x_n < \lambda \end{cases}$$

As previous theorems, we obtain a constant $\gamma_4 > 1$, independent of $u(x)$, such that

$$\int_{\{x \in S_1 : x_n > \lambda\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \le \gamma_4 \left\{ \int_{\{x \in S_1 : \lambda < x_n < \lambda + \lambda^m\}} \sum_{i=1}^n |\theta_\lambda(x_n)|^{p+1} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx + \right.$$

$$\left. + \int_{\{x \in S_1 : \lambda < x_n < \lambda + \lambda^m\}} |u(x)|^{p+1} |\theta'_\lambda(x_n)|^{p+1} dx \right\} \le$$

$$\le \gamma_4 \left\{ \int_{\{x \in S_1 : \lambda < x_n < \lambda + \lambda^m\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} + \lambda^{-m(p+1)} |u(x)|^{p+1} \right\} dx, \text{ for } \lambda \text{ large enough.}$$

From this inequality and Friedrichs inequality, we obtain

$$\int_{\{x \in S_1 : x_n > \lambda\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \le \gamma_4 c_9 \int_{\{x \in S_1 : \lambda < x_n < \lambda + \lambda^m\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

Next, a simple computation gives

$$\int_{\{x \in S_1 : x_n > \lambda + \lambda^m\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \le$$

(3.13)

$$\le \left(1 - \frac{1}{\gamma_4 c_{10}} \right) \int_{\{x \in S_1 : x_n > \lambda\}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx,$$

for λ large enough; $c_{10} = c_9 + 1$.

Then, for t large enough we have

$$\int_{\{x\in S: x_n>2^t\}} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx \le A\left(1-\frac{1}{\gamma_4 c_{10}}\right)^{t^{1-m}}.$$

We conclude our theorem with another application of Friedrichs inequality.

Remark (3.7). From (3.13) it also follows that:

$$\int_{\{x\in S: x_n>t\}} \left(|u|^{p+1} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1}\right) dx \le c_{11} t^{-\delta(1-m)}, \ \forall\, t \ \text{ large enough.}$$

Finally, we consider solutions of equation (1.1), in unbounded domain Q, for which the condition $\int_Q \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx < +\infty$ cannot be verified. What happens, for instance, in the cylindrical domain π_0?

We shall show that it holds the following

Theorem (3.8) *Let $u(x)$ be a solution of* (1.1) *in π_0 with homogeneous Dirichlet data on σ_0 and, moreover, such that*

$$\int_{\pi_0} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} e^{\beta x_n} dx < +\infty,$$

for some $\beta < 0$. Then, there exists a positive constant $\epsilon(\Omega) > 0$ such that if $|\beta| \le \epsilon(\Omega)$

$$\int_{\pi_0} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx < +\infty.$$

Proof.- Let $t > 2$ and assume that $f(x) = 0$ in π_0; we introduce real functions $\theta(x_n) \in C^\infty(\mathbb{R})$, $\beta(x_n)$ by

$$\theta(x_n) = \begin{cases} 0 & if \quad 0 < x_n < 1 \\ 1 & if \quad x_n > 2 \end{cases},$$

$$\beta(x_n) = \begin{cases} \beta x_n & if \quad x_n > t \\ \beta t & if \quad x_n \le t \end{cases}$$

Multiplying equation (1.1) by $[\theta(x_n)e^{\beta(x_n)}u - \epsilon]$ with ϵ small enough, integrating it over π_0 , we have (letting ϵ to zero)

$$\int_{\pi_0} \sum_{i=1}^{n} \frac{\partial\left[\theta(x_n)e^{\beta(x_n)}u\right]}{\partial x_i} \frac{\partial u}{\partial x_i} \left|\frac{\partial u}{\partial x_i}\right|^{p-1} dx = 0.$$

From this, we get

$$\text{(3.14)} \qquad \int_{\pi_0} \sum_{i=1}^{n} \theta(x_n)e^{\beta(x_n)} \left|\frac{\partial u}{\partial x_i}\right|^{p+1} dx \le \int_{\pi_0} \left|\frac{\partial\left[\theta(x_n)e^{\beta(x_n)}\right]}{\partial x_n}\right| |u| \left|\frac{\partial u}{\partial x_n}\right|^{p} dx.$$

Now, the left-side term of (3.14) can be estimate in this way

$$\int_{\pi_0} \sum_{i=1}^{n} \theta(x_n) e^{\beta(x_n)} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \geq \sum_{i=1}^{n} \int_{\pi_{2,t}} e^{\beta t} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx + \int_{\pi_t} e^{\beta x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

On the other hand, for the right-side term of (3.14) we have

$$\int_{\pi_0} \left| \frac{\partial \left[\theta(x_n) e^{\beta(x_n)}\right]}{\partial x_n} \right| |u| \left| \frac{\partial u}{\partial x_n} \right|^p dx \leq e^{\beta t} \int_1^2 \int_\Omega |\theta'(x_n)||u| \left| \frac{\partial u}{\partial x_n} \right|^p dx +$$
$$+ \int_t^{+\infty} \int_\Omega |\beta||\theta(x_n)| e^{\beta x_n} |u| \left| \frac{\partial u}{\partial x_n} \right|^p dx \leq$$
$$\leq e^{\beta t} \int_{\pi_{1,2}} \sum_{i=1}^{n} |u| \left| \frac{\partial u}{\partial x_i} \right|^p dx + \int_{\pi_t} \sum_{i=1}^{n} |\beta||\theta(x_n)| e^{\beta x_n} |u| \left| \frac{\partial u}{\partial x_i} \right|^p dx.$$

From (3.14), taking into account last two inequalities, we get

$$\sum_{i=1}^{n} \int_{\pi_{2,t}} e^{\beta t} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx + \int_{\pi_t} e^{\beta x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq$$
$$\leq B e^{\beta t} + \sum_{i=1}^{n} |\beta| \left(\int_{\pi_t} e^{\beta x_n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \right)^{\frac{p}{p+1}} \left(\int_{\pi_t} e^{\beta x_n} |u|^{p+1} dx \right)^{\frac{1}{p+1}}, \tag{3.15}$$

where $B = \int_{\pi_{1,2}} \sum_{i=1}^{n} |u| \left| \frac{\partial u}{\partial x_i} \right|^p dx$.

According to (F), we have

$$\int_{\pi_t} e^{\beta x_n} |u(x)|^{p+1} dx \leq c(n, p, \Omega) \int_{\pi_t} e^{\beta x_n} \sum_{i=1}^{n-1} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

This fact applied to (3.15) gives

$$\sum_{i=1}^{n} \int_{\pi_{2,t}} e^{\beta t} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx + \int_{\pi_t} e^{\beta x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq$$
$$\leq B e^{\beta t} + n \sum_{i=1}^{n} |\beta| c_{12} \int_{\pi_t} e^{\beta x_n} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx.$$

Furthermore, if

$$|\beta| \leq \epsilon(\Omega) = \frac{1}{n c_{12}}$$

it follows

$$\sum_{i=1}^{n} \int_{\pi_{2,t}} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq B.$$

An other application of Friedrichs inequality gives

$$B \leq n c_{13} I_{1,2}(u)$$

and so,

$$\sum_{i=1}^{n} \int_{\pi_{2,t}} \left| \frac{\partial u}{\partial x_i} \right|^{p+1} dx \leq c_{14} I_{1,2}(u), \text{ for } t > 2.$$

Letting t to infinity we have our assertion.

4 Conclusion

We finally note that it is possible to extend Theorems (3.1) and (3.4) to solutions of the following nonlinear equation

$$\text{div } a(Du) - c_0 |u|^{p-1} u = f(x) \text{ in } Q \subset \mathbb{R}^n,$$

where c_0 is a nonnegative constant and the vector field $a : \mathbb{R}^n \to \mathbb{R}^n$, assumed to be C^1-regular, satifies the following growth and ellipticity assumptions

$$\text{(4.1)} \qquad \begin{cases} |a(z)| + |a_z(z)||z| \leq L|z|^p \\ \nu |z|^{p-1} |\xi|^2 \leq \langle a_z(z)\xi, \xi \rangle, \end{cases}$$

whenever $z, \xi \in \mathbb{R}^n$, $p \geq 1$ and, $0 < \nu \leq L$ are fixed parameters. In such case, it will be important to note that $(4.1)_b$ implies the existence of a positive constant $\tilde{c} = \tilde{c}(n, p, \nu) > 1$ such that the following inequality holds whenever $z_1, z_2 \in \mathbb{R}^n$

$$\tilde{c}^{-1} |z_2 - z_1|^{p+1} \leq \langle a(z_2) - a(z_1), z_2 - z_1 \rangle.$$

A model case for the previous situation is clearly given by considering the p-Laplacean equation (1.1).

Competing Interests

Author has declared that no competing interests exist.

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