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Irredundant and Almost Irredundant Sets in $\mathbb{M}_2(\mathbb{C})$

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The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

We consider irredundant and almost irredundant subsets in the *-algebra $\mathbb{M}_2(\mathbb{C})$ of all 2×2 matrices with coefficients in $\mathbb{C}$. We prove that the largest size of an irredundant subset is two, and that $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant subset.

Keywords: Matrix algebra; generators; irredundancy; involution.

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1 Introduction

Let $\mathbb{M}_2(\mathbb{C})$ represent the algebra of all 2×2 matrices with coefficients in $\mathbb{C}$. For a given subset $S \subset \mathbb{M}_2(\mathbb{C})$, denote by $alg(S)$ the unital subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by S. A natural question that arises is under what conditions $alg(S) = \mathbb{M}_2(\mathbb{C})$; in other words, when can a subset S generate the entire algebra $\mathbb{M}_2(\mathbb{C})$? Furthermore, an interesting problem is to determine the largest size of a set S that can generate $\mathbb{M}_2(\mathbb{C})$, while

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ensuring that no proper subset of S possesses this generating property.
These questions have been the subject of extensive research in matrix theory. Investigations into when a subset generates the full matrix algebra are detailed in [1]. Additionally, T. Laffey addresses the problem of determining the maximum size of an irredundant set of generators in [2].

In this study, we also consider the algebra $\mathbb{M}_2(\mathbb{C})$ equipped with an involution $^* : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})$ defined by the conjugate transpose, $A^* = \overline{A}^t$. With this operation, $\mathbb{M}_2(\mathbb{C})$ becomes a *-algebra (or an involutive algebra). For a subset $S \subset \mathbb{M}_2(\mathbb{C})$, we denote by $alg^*(S)$ the involutive unital subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by S. If $alg^*(S) = \mathbb{M}_2(\mathbb{C})$, we say that S is a set of *-generators for $\mathbb{M}_2(\mathbb{C})$, or that S *-generates $\mathbb{M}_2(\mathbb{C})$.

Again, we can inquire about the maximum size[1] of a *-generator S for $\mathbb{M}_2(\mathbb{C})$, ensuring that no proper subset of S is also a set of *-generators for $\mathbb{M}_2(\mathbb{C})$. This property is referred to as 'irredundancy'.

Definition 1.1. Let S be a subset of $\mathbb{M}_2(\mathbb{C})$. We say that S is irredundant, if $x \notin alg(S \setminus \{x\})$ for every $x \in S$. Moreover, if $x \notin alg^*(S \setminus \{x\})$, for every $x \in S$, we say that S is a *-irredundant set.

In other words, a subset $S \subset \mathbb{M}_2(\mathbb{C})$ is considered irredundant (*-irredundant) if no element of S is contained in the (involutive) subalgebra generated by the other elements in S.

As a consequence of Laffey's result [2, Theorem 2.1], we demonstrate in Section 2 that the maximum size of a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$ is two.

The notion of *-irredundance in general infinite dimensional C*-algebras has been introduced in [3], where the question whether every C*-algebra has a large *-irredundant set was considered [4]. In an attempt to prove the existence of large *-irredundant sets, a weaker notion of *-irredundance, termed almost irredundance, was introduced in [5]. It was demonstrated that a special class of C*-algebras admits large almost irredundant sets (see [5] for details).

In this article, we establish that in the finite-dimensional context, specifically for the algebra $\mathbb{M}_2(\mathbb{C})$, the maximal size of an irredundant sets and almost irredundant sets differ significantly. In Section 3, we show that $\mathbb{M}_2(\mathbb{C})$ possesses an infinite almost irredundant set (see Proposition 3.1).

2 Irredundant Sets in $\mathbb{M}_2(\mathbb{C})$

By definition, *-irredundancy implies irredundancy, and every irredundant set is, in particular, a linearly independent set. Consequently, the size of a *-irredundant set in $\mathbb{M}_2(\mathbb{C})$ is bounded above by four.

Consider a subset S of $\mathbb{M}_2(\mathbb{C})$. Suppose $|S| = 1$. Then, $alg(S)$ is a commutative algebra, implying $alg(S) \neq \mathbb{M}_2(\mathbb{C})$. This demonstrates the absence of any $S \subset \mathbb{M}_2(\mathbb{C})$ of size 1 capable of generating the entire algebra $\mathbb{M}_2(\mathbb{C})$.

Now, let's suppose $|S| = 2$. According to Burnside's theorem[2], a subset $S = \{A_1, A_2\}$ generates $\mathbb{M}_2(\mathbb{C})$ if A_1 and A_2 do not share a common eigenvector. Particularly, if $E_{i,j}$ denotes the 2×2 matrix with a one in the (i, j)-position and zero elsewhere, then $S = \{E_{1,2}, E_{2,1}\}$ constitutes an irredundant set of generators of size 2 for the algebra $\mathbb{M}_2(\mathbb{C})$. Thus, the smallest possible size of a set of irredundant generators for $\mathbb{M}_2(\mathbb{C})$ as an algebra is 2.

If we consider an involution, then $A = E_{1,2}$ satisfies

$$\{A, A^*, AA^*, A^*A\} = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}.$$

[1] For details on the problem of finding generating sets in matrix algebras with generic involutive maps, see [6].

[2] We recommend [7] for a concise proof of Burnside's theorem.

This demonstrates that $S = \{A\}$ is a *-irredundant set (due to its sole member), and $alg^*(S) = \mathbb{M}_2(\mathbb{C})$. Particularly, when an involution is incorporated in our operations, the lower bound of an irredundant set of generators for the algebra $\mathbb{M}_2(\mathbb{C})$, which is two, reduces to one.

According to [2, Theorem 2.1], the maximum size of an irredundant set of generators for $\mathbb{M}_2(\mathbb{C})$ is three. We establish that upon integrating an involution into our operations, the upper bound is similarly reduced by one unit. In other words, we demonstrate that the maximum size of a *-irredundant set that *-generates $\mathbb{M}_2(\mathbb{C})$ is two. Before delving into the proof, we introduce some auxiliary lemmas.

Recall that a matrix $A \in \mathbb{M}_2(\mathbb{C})$ is self-adjoint if $A^* = A$, and unitary if $AA^* = A^*A = Id$. Each matrix can be expressed as a linear combination of two self-adjoint elements. For every $A \in \mathbb{M}_2(\mathbb{C})$, we denote $A = B + iC$, where $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$ are self-adjoint matrices.

The following lemma[3] states that elements in a *-irredundant set can be replaced with self-adjoint elements, resulting in another *-irredundant set.

Lemma 2.1. *Let F be a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$ of size n, where n represents the largest size of such a set. Then, there exists a *-irredundant set of *-generators F' of size n composed entirely of self-adjoint elements.*

Proof. Let $F = \{A_1, A_2, \cdots, A_n\}$ be a *-irredundant set. Write $A_1 = B_1 + iC_1$, where $B_1 = \frac{1}{2}(A_1 + A_1^*)$ and $C_1 = \frac{1}{2i}(A_1 - A_1^*)$ are self-adjoint elements. If $B_1, C_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$, then $A_1 = B_1 + iC_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$, which contradicts the fact that $\{A_1, A_2, \cdots, A_n\}$ is a *-irredundant set.

Claim 1. It is always possible to choose $D \in \{B_1, C_1\}$ such that $\{D, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$ and $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$.

In fact, since $\{A_1, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$, it is sufficient to show that we can choose D such that $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$ and $A_1 \in alg^*(\{D, A_2, A_3, \cdots, A_n\})$.

If $B_1 \in alg^*(\{C_1, A_2, A_3, \cdots, A_n\})$, choose $D = C_1$. Observe that $D = C_1 \notin alg^*(\{A_2, A_3, \cdots, A_n\})$, otherwise we would have $B_1, C_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$ and therefore,

$$A_1 = B_1 + iC_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$$

which is a contradiction with the irredundancy of F.

Suppose now that $B_1 \notin alg^*(\{C_1, A_2, A_3, \cdots, A_n\})$ and define $D = B_1$. We claim that $C_1 \in alg^*(\{D, A_2, A_3, \cdots, A_n\})$; In fact, since

$$alg^*(\{B_1, C_1, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\}) = alg^*(\{A_1, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\})$$

for every $2 \leq j \leq n$, we would have that $\{B_1, C_1, A_2, \cdots, A_n\}$ is a *-irredundant set which *-generates $\mathbb{M}_2(\mathbb{C})$ and contains $n + 1$ elements, which contradicts the choice of n.

Fix $D \in \{B_1, C_1\}$ as in Claim 1. Then $\{D, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$. Let us proove that $\{D, A_2, A_3, \cdots, A_n\}$ is a *-irredundant set. Since $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$ it sufficies to show that there is no $2 \leq j \leq n$ such that

$$A_j \in alg^*(\{D, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\})$$

[3] For a version of this lemma applicable to all C*-algebras, refer to [3, Proposition 3.2].

Suppose there exists such j and let us derive a contradiction. Since $D \in alg^*(A_1)$ it follows that $alg^*(D, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n)$ is a subset of $alg^*(\mathcal{F} \setminus \{A_j\})$ and therefore, $A_j \in alg^*(\mathcal{F} \setminus \{A_j\})$, contradicting the *-irredundancy of $\mathcal{F}$. In particular, $\{D, A_2, A_3, \cdots, A_n\}$ is an *-irredundant set of *-generators comprising self-adjoint elements. □

Now, using the fact that we are only working in the field of complex numbers, we can rewrite [2, Theorem 2.1] as follows:

Proposition 2.1 ([2, Theorem 2.1])**.** *Let $S \subset \mathbb{M}_2(\mathbb{C})$ be an irredundant set of self-adjoint elements such that $alg(S) = \mathbb{M}_2(\mathbb{C})$. Then $|S| \leq 3$. Moreover, if $|S| = 3$, then there is a unitary $U \in \mathbb{M}_2(\mathbb{C})$ such that $U^*SU = \{A, B, C\}$, where $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \alpha I_2$, $B = \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} + \beta I_2$ and $C = \begin{pmatrix} d & 0 \\ e & 0 \end{pmatrix} + \gamma I_2$, where $a, b, c, d, \alpha, \beta, \gamma \in \mathbb{C}$, with $a \neq 0$ and $bd + ce = 0$.*

Proposition 2.2. *Let n be the largest possible size of a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$. Then $n \leq 2$.*

Proof. Suppose $S \subset \mathbb{M}_2(\mathbb{C})$ is an *-irredundant set which *-generates $\mathbb{M}_2(\mathbb{C})$ as involutive algebra with the largest possible size. From Lemma 2.1, we can assume that all the elements in S are self-adjoint elements. As S is formed by self-adjoint elements, $alg^*(S) = alg(S)$. Then, S is a *-irredundant set (and therefore irredundant) which generates $\mathbb{M}_2(\mathbb{C})$ as an algebra. In particular, from Proposition 2.1, we have $|S| \leq 3$. Assume that $S = \{A_1, A_2, A_3\}$ and lets get a contradiction. By Proposition 2.1, there is an unitary U such that $U^*SU = \{A, B, C\}$, where

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \alpha I_2,\ B = \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} + \beta I_2 \text{ and } C = \begin{pmatrix} d & 0 \\ e & 0 \end{pmatrix} + \gamma I_2 \text{ for some } a, b, c, d, \alpha, \beta, \gamma \in \mathbb{C}.$$

Since S if formed by self-adjoint elements and U is unitary, the matrices A, B, C are all diagonal matrices. In particular, $\{A, B, C\}$ are linearly dependent. Since the map $a \to U^*aU$ defines a bijective involutive morphism, it follows that $\{A_1, A_2, A_3\}$, should be linearly dependent, which contradicts the fact that $S = \{A_1, A_2, A_3\}$ is a *-irredundant set. □

The following remark shows that the upper bound for *-irredundant set is attained:

Consider $\mathcal{F} = \{A, B\}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Observe that A and B have no commom-invariant subspaces. Then, by Burside's theorem, $\{A, B\}$ generates $\mathbb{M}_2(\mathbb{C})$ (as an algebra and, in particular, as an involutive algebra). In conclusion, we observe that $A \notin alg^*(B)$ and $B \notin alg^*(A)$, which shows that $\{A, B\}$ is a *-irredundant set.

3 Almost Irredundant Sets in $\mathbb{M}_2(\mathbb{C})$

We focus now on a weaker notion of *-irredundance introduced in [5]. Let $S \subset \mathbb{M}_2(\mathbb{C})$ be a self-adjoint subset of $\mathbb{M}_2(\mathbb{C})$. Then, S is *-irredundant if and only if for every $a \in S$, the element a does not belong to the involute subalgebra generated by $S \setminus \{a\}$. That is, a cannot be written as $\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} a_{i,j}$, where $a_{i,j} \in S \setminus \{a\}$ and $\lambda_i \in \mathbb{C}$.

Let us restrict the coefficients λ's and define the following weak notion of *-irredundance:

Definition 3.1. Let S be a subset of $\mathbb{M}_2(\mathbb{C})$. Then, S is almost irredundant if and only if, for every $a \in S$, the element a cannot be written as $\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} a_{i,j}$, where $a_{i,j} \in S \setminus \{a\}$ and $\sum_{i=1}^{n} |\lambda_i| \leq 1$.

Observe that the main difference in the definition of *-irredundant sets and almost irredundant sets is that in the first, we allow any linear combinations, whereas in the second, we allow only convex linear combinations. In particular, any *-irredundant set is an almost irredundant set. However, we will see that these two notions behave differently when we consider the maximal size of such sets. We will prove that $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant set.

First, some lemmas are required. Recall that a self-adjoint matrix $A \in \mathbb{M}_2(\mathbb{C})$ is positive if its spectrum is positive, and that a linear map $\tau : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ is positive if $\tau(A) \geq 0$ whenever $A \in \mathbb{M}_2(\mathbb{C})$ is positive. We say that a matrix $A \in \mathbb{M}_2(\mathbb{C})$ is a projection if A is self-adjoint and $A^2 = A$. Given a matrix $A \in \mathbb{M}_2(\mathbb{C})$, consider $\|A\|$ as the operator norm of A.

Lemma 3.1. *Let $\tau : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be a positive map and $A_1, \cdots, A_n \in \mathbb{M}_2(\mathbb{C})$. Then*

$$\tau(A_n^* \cdots A_1^* A_1 \cdots A_n) \leq \|A_1\|^2 \cdots \|A_{n-1}\|^2 \tau(A_n^* A_n)$$

Proof. The proof of the lemma follows from repeatedly applying the inequality $\tau(B^*A^*AB^*) \leq \|A^*A\|\tau(B^*B)$ that holds for every $A, B \in \mathbb{M}_2(\mathbb{C})$ (see Theorem 3.3.7 of [8]). □

Lemma 3.2. *Let $P \in \mathbb{M}_2(\mathbb{C})$ be a projection, and let $\tau : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be the map defined by $\tau(A) = trace(PA)$ for every $A \in \mathbb{M}_2(\mathbb{C})$, where $trace : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ is the canonical trace of a matrix. Let $A_1, \cdots, A_n$ be projections such that $\tau(A_i) < 1$ for every $1 \leq i \leq n$. Then*

$$\tau(A_1 A_2 \cdots A_n) < 1.$$

Proof. First, we observe that the linear map τ is positive, therefore, the map $(A, B) \to \tau(B^*A)$ defines a positive sesquilinear form on $\mathbb{M}_2(\mathbb{C})$. In particular, we apply the Cauchy–Schwarz inequality to show that $|\tau(B^*A)|^2 \leq \tau(B^*B)\tau(A^*A)$. Now, using Lemma 3.1 and the fact that $\tau(A_1), \tau(A_n) < 1$ we obtain:

$$\begin{aligned}
|\tau(A_1 A_2 \cdots A_n)|^2 &= |\tau((A_1)(A_2 \cdots A_n))|^2 \\
&\leq \tau(A_1^* A_1)\tau((A_2 \cdots A_n)^*(A_2 \cdots A_n)) \\
&\leq \tau(A_1)\tau(A_n^* \cdots A_2^* A_2 \cdots A_n) \\
&< \tau(A_n^* \cdots A_2^* A_2 \cdots A_n) \\
&\leq \|A_2\|^2 \cdots \|A_{n-1}\|^2 \tau(A_n^* A_n)) \\
&\leq \tau(A_n^* A_n) = \tau(A_n) \\
&< 1
\end{aligned}$$

□

Lemma 3.3. *Consider the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ given by*

$$f(x, y) = (xy + \sqrt{1 - x^2}\sqrt{1 - y^2})^2.$$

Then, there exists an infinite family of distinct points $(x_i)_{i \in \mathbb{N}}$ such that:

1. *$f(x_i, x_i) = 1$ for every $i \in \mathbb{N}$;*
2. *$f(x_i, x_j) < 1$ for every $i \neq j \in \mathbb{N}$.*

Proof. Consider a sequence of distinct points $(\theta_n)_{n \in \mathbb{N}}$ in $[0, \pi/2]$ and define $x_n = \cos(\theta_n)$ for each $n \in \mathbb{N}$. We claim that the family of points $(x_n)_{n \in \mathbb{N}}$ has the desirable properties: one has:

$$\begin{aligned}
F(x_i, x_j) &= \cos(\theta_i)\cos(\theta_j) + \sin(\theta_i)\sin(\theta_j) \\
&= \cos(\theta_i - \theta_j)
\end{aligned}$$

It follows that $f(x_i, x_j) < 1$ when $i \neq j$ and $f(x_i, x_i) = 1$ for each $i, j \in \mathbb{N}$ as required. □

Proposition 3.1. $\mathbb{M}_2(\mathbb{C})$ *has an infinite almost irredundant set (of projections).*

Proof. Fix $(x_i)_{i\in\mathbb{N}}$ given by Lemma 3.3. For each $i \in \mathbb{N}$, define $y_i = \sqrt{1-x_i^2}$ and the orthogonal projection onto the vector $v_{x_i} = (x_i, y_i)$ given by the matrix $A_i = \begin{pmatrix} x_i^2 & x_iy_i \\ x_iy_i & y_i^2 \end{pmatrix}$. Let $\tau_i : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be a linear map defined as $\tau_i(A) = trace(A_iA)$. Given $i, j \in \mathbb{N}$ we have that

$$\begin{aligned}
\tau_i(A_j) &= trace\left(\begin{pmatrix} x_i^2 & x_iy_i \\ x_iy_i & y_i^2 \end{pmatrix}\begin{pmatrix} x_j^2 & x_jy_j \\ x_jy_j & y_j^2 \end{pmatrix}\right) \\
&= trace\left(\begin{pmatrix} x_i^2x_j^2 + x_iy_ix_jy_j & \cdots \\ \cdots & y_i^2y_j^2 + x_iy_ix_jy_j \end{pmatrix}\right) \\
&= x_i^2x_j^2 + y_i^2y_j^2 + 2x_iy_ix_jy_j \\
&= (x_ix_j + y_iy_j)^2 \\
&= (x_ix_j + \sqrt{1-x_i^2}\sqrt{1-x_j^2})^2 \\
&= f(x_i, x_j)
\end{aligned}$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$ is the function from Lemma 3.3. In particular, by Lemma 3.3 we have that

1. $\tau_i(A_i) = 1$ and
2. $\tau_i(A_j) < 1$ if $i \neq j$.

Let us prove that $(A_i)_{i\in\mathbb{N}}$ is an almost irredundant set. Suppose by contradiction that $(A_i)_{i\in\mathbb{N}}$ is not an almost irredundant set. Without loss of generality, suppose that we can write $A_1 = \sum_{i=1}^m \lambda_i \prod_{j=1}^{n_i} a_{i,j}$ where $a_{i,j} \neq A_1$ and $\sum_{i=1}^m |\lambda_i| \leq 1$. By Lemma 3.2 we conclude that

$$\begin{aligned}
1 &= |\tau_1(A_1)| \\
&= |\tau_1(\sum_{i=1}^m \lambda_i \prod_{j=1}^{n_i} a_{i,j})| \\
&\leq \sum_{i=1}^m |\lambda_i||\tau_1(\prod_{j=1}^{n_i} a_{i,j})| \\
&< \sum_{i=1}^m |\lambda_i| \\
&\leq 1
\end{aligned}$$

which is a contradiction. □

4 Conclusions

The notion of *-irredundance in general infinite dimensional C*-algebras has been introduced in [3] and it is defined in an analogous manner as for matrix algebras. Because every C*-algebra is in particular a Banach space, every infinite-dimensional C*-algebra has an uncountable linear dimension; therefore, some other cardinals are more appropriate to tell something about the “size” of the algebra. For instance, the topological density of the algebra. Then, we can ask whether every large C*-algebra (in the sense of big density) has a large *-irredundant set. Some answers to this question have some set-theoretic flavours in the sense that we need to add some extra set-theoretic axioms to the standard ZFC axioms. One of the fundamental results is the example of a commutative C*-algebra with a larger density without uncountable *-irredundant sets, which is obtained as a C*-algebra of the form $C(K)$, where K is the Kunen space obtained under the Continuum Hypothesis (see [9]). The question of whether such an example exists in ZFC remains open. An important partial result in this direction is the result of Todorcevic (see [10, 11]). We refer the reader to [3] for further details on *-irredundant sets in C*-algebras.

The notion of an almost irredundant set was introduced in [5] in an attempt to answer questions on *-irredundant sets. In particular, we mention [5, Theorem 1.3], where the author proved that it is consistent with the ZFC that large C*-algebras of some special class of C*-algebras admit an uncountable, almost irredundant set. Also, we refer the reader to [4] for some cardinal inequalities for almost irredundant sets.

In this article, we have proved that the maximal size of a *-irredundant set in $\mathbb{M}_2(\mathbb{C})$ is 2, while $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant set. In the infinite dimensional case, every infinite-dimensional C*-algebra has an infinite *-irredundant set (see [3, Proposition 3.12]). It is an open question whether the maximal size of a *-irredundant set is equal to the size of an almost irredundant set for an infinite dimensional C*-algebra. In particular, it is open if there can be a nonseparable C*-algebra, with an uncountable almost irredundant set, and with no uncountable *-irredundant set.

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