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A Study on Matrix Sequence with Generalized Guglielmo Numbers Components

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

We know that the diverse applications of matrix sequences in fields such as physics, engineering, architecture, nature, and art. Numerous authors have delved into the study of these matrix sequences in existing literature. In this study, we define and investigate the generalized Guglielmo matrix sequence. For this aim we explore four specific cases of that sequence that are called triangular matrix sequences, Lucas-triangular matrix sequences, oblong matrix sequences, and pentagonal matrix sequences. Next, we present Binet's formulas, generating functions, the summation formulas and some elementary identities for these sequences. Moreover,

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we give some identities and matrices related with these sequences. Furthermore, we show that there always exist some relationship between generalized triangular, Lucas-triangular, oblong and pentagonal matrix sequences.

Keywords: *Triangular matrix sequence; Lucas-triangular matrix sequence, oblong matrix sequence; pentagonal matrix sequence.*

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1 Introduction

In this section, we present some properties of the generalized Guglielmo sequence, such as reccurance relations, Binet's formula, generating function and characteristic equations, that we will need rest of the our study. For more detail see, Soykan, (2022).

A generalized Guglielmo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is given by the third-order recurrence relations.

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Now we present four special cases of the sequence $\{W_n\}$. Triangular sequence $\{T_n\}_{n\geq 0}$, triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentagonal sequence $\{p_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
T_n &= 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0,\ T_1 = 1,\ T_2 = 3, && (1.2)\\
H_n &= 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3,\ H_1 = 3,\ H_2 = 3, && (1.3)\\
O_n &= 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0,\ O_1 = 2,\ O_2 = 6, && (1.4)\\
p_n &= 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0,\ p_1 = 1,\ p_2 = 5. && (1.5)
\end{aligned}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)},\\
H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)},\\
O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)},\\
p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)}
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.5) hold for all integer n. Now, we give some properties related to generalized Guglielmo numbers that we need for the rest of the study.

- The Binet's formula of generalized Guglielmo numbers can be given as

$$W_n = A_1 + A_2 n + A_3 n^2 \tag{1.6}$$

where

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0), \end{aligned}$$

i.e.,

$$W_n = W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2. \tag{1.7}$$

- For all integers n, triangular, triangular-Lucas, oblong and pentagonal numbers (using initial conditions in (1.7)) can be expressed using Binet's formulas as

$$\begin{aligned} T_n &= \frac{n(n+1)}{2}, \\ H_n &= 3, \\ O_n &= n(n+1), \\ p_n &= \frac{1}{2}n(3n-1) \end{aligned}$$

respectively.

- Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Guglielmo sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}. \tag{1.8}$$

- Here, the characteristic equation of the generalized Guglielmo sequence

$$x^3 - 3x^2 + 3x - 1 = 0.$$

- (Simpson's formula for generalized Guglielmo numbers) For all integers n, we have the following identity.

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -(W_2 - 2W_1 + W_0)^3.$$

Next, the first few generalized Guglielmo numbers with positive subscript and negative subscript is given in the following Table 1.

Next, we present the first few values of the Triangular and Triangular-Lucas, oblong and pentagonal numbers with positive and negative subscripts:

For more detail, see Soykan, (2022).

Table 1. A few generalized guglielmo numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$
7	$15W_0 - 35W_1 + 21W_2$	$36W_0 - 63W_1 + 28W_2$
8	$21W_0 - 48W_1 + 28W_2$	$45W_0 - 80W_1 + 36W_2$
9	$28W_0 - 63W_1 + 36W_2$	$55W_0 - 99W_1 + 45W_2$
10	$36W_0 - 80W_1 + 45W_2$	$66W_0 - 120W_1 + 55W_2$
11	$45W_0 - 99W_1 + 55W_2$	$78W_0 - 143W_1 + 66W_2$
12	$55W_0 - 120W_1 + 66W_2$	$91W_0 - 168W_1 + 78W_2$
13	$66W_0 - 143W_1 + 78W_2$	$105W_0 - 195W_1 + 91W_2$

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	3	6	10	15	21	28	36	45	55	66	78	91
T_{-n}		0	1	3	6	10	15	21	28	36	45	55	66	78
H_n	3	3	3	3	3	3	3	3	3	3	3	3	3	3
H_{-n}		3	3	3	3	3	3	3	3	3	3	3	3	3
O_n	0	2	6	12	20	30	42	56	72	90	110	132	156	182
O_{-n}		0	2	6	12	20	30	42	56	72	90	110	132	156
p_n	0	1	5	12	22	35	51	70	92	117	145	176	210	247
p_{-n}		2	7	15	26	40	57	77	100	126	155	187	222	260

2 The Matrix Sequences of Generalized Guglielmo Numbers

Recently, numerous studies have explored sequences of numbers in the literature focusing on sequences of Horadam (generalized Fibonacci) numbers and generalized Tribonacci numbers. These include well-known sequences such as Fibonacci, Lucas, Pell, and Jacobsthal numbers, as well as third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third-order Jacobsthal and third-order Jacobsthal-Lucas numbers. The sequences of numbers have found widespread applications in various research areas, including physics, engineering, architecture, nature and art. Conversely, matrix sequences have garnered attention, particularly for different types of numbers. Next, we present some study published in the litherature related to matrix sequences.

- For study related to Generalized Fibonacci matrix sequences see, Civciv and Turkmen, (2008a),Civciv and Turkmen, (2008b),Frontczak, (2018), Gulec and Taskara, (2012), Uslu and Uygun Ş, (2013), Uygun Ş and Uslu K, (2015), Uygun Ş, (2016), Uygun Ş, (2019), Yazlik et al., (2012), Wani et al., (2018).
- For study related to Generalized Tribonacci matrix sequences see, Cerda-Morales, (2019), Soykan, (2020), Soykan, (2020), Soykan et al., (2021), Soykan et al., (2021), Yilmaz et al., (2013), Yilmaz et al., (2014).
- For study related to Generalized Tetranacci matrix sequences see, Soykan, (2019).

In this section we define generalized Guglielmo matrix sequence and investigate its properties.

Definition 2.1. For any integer $n \geq 0$, the generalized Guglielmo matrix $(\mathcal{M}W_n)$ is defined by

$$\mathcal{M}W_n = 3\mathcal{M}W_{n-1} - 3\mathcal{M}W_{n-2} + \mathcal{M}W_{n-3} \tag{2.1}$$

with initial conditions

$$\begin{aligned}
\mathcal{M}W_0 &= \begin{pmatrix} W_1 & W_2 - 3W_1 & W_0 \\ W_0 & W_1 - 3W_0 & 3W_0 - 3W_1 + W_2 \\ 3W_0 - 3W_1 + W_2 & 9W_1 - 8W_0 - 3W_2 & 6W_0 - 8W_1 + 3W_2 \end{pmatrix}, \\
\mathcal{M}W_1 &= \begin{pmatrix} W_2 & W_0 - 3W_1 & W_1 \\ W_1 & W_2 - 3W_1 & W_0 \\ W_0 & W_1 - 3W_0 & 3W_0 - 3W_1 + W_2 \end{pmatrix}, \\
\mathcal{M}W_2 &= \begin{pmatrix} W_0 - 3W_1 + 3W_2 & W_1 - 3W_2 & W_2 \\ W_2 & W_0 - 3W_1 & W_1 \\ W_1 & W_2 - 3W_1 & W_0 \end{pmatrix}.
\end{aligned}$$

The sequence $\{\mathcal{M}W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{M}W_{-n} = 3\mathcal{M}W_{-(n-1)} - 3\mathcal{M}W_{-(n-2)} + \mathcal{M}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.1) holds for all integers n.

Three special cases of generalized Guglielmo matrix sequence, taking $W_n = T_n, W_n = H_n, W_n = O_n, W_n = p_n$, respectively, can be defined as follows.

Definition 2.2. For any integer $n \geq 0$, the triangular matrix $(\mathcal{M}T_n)$, Lucas-triangular matrix $(\mathcal{M}H_n)$, oblong matrix $(\mathcal{M}O_n)$ and pentagonal matrix $(\mathcal{M}p_n)$ are defined by

$$\begin{aligned}
\mathcal{M}T_n &= 3\mathcal{M}T_{n-1} - 3\mathcal{M}T_{n-2} + \mathcal{M}T_{n-3}, \\
\mathcal{M}H_n &= 3\mathcal{M}H_{n-1} - 3\mathcal{M}H_{n-2} + \mathcal{M}H_{n-3}, \\
\mathcal{M}O_n &= 3\mathcal{M}O_{n-1} - 3\mathcal{M}O_{n-2} + \mathcal{M}O_{n-3}, \\
\mathcal{M}p_n &= 3\mathcal{M}p_{n-1} - 3\mathcal{M}p_{n-2} + \mathcal{M}p_{n-3}
\end{aligned}$$

respectively, with initial conditions

$$\begin{aligned}
\mathcal{M}T_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathcal{M}T_1 = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathcal{M}T_2 = \begin{pmatrix} 6 & -8 & 3 \\ 3 & -3 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}H_0 &= \begin{pmatrix} 3 & -6 & 3 \\ 3 & -6 & 3 \\ 3 & -6 & 3 \end{pmatrix}, \ \mathcal{M}H_1 = \begin{pmatrix} 3 & -6 & 3 \\ 3 & -6 & 3 \\ 3 & -6 & 3 \end{pmatrix}, \ \mathcal{M}H_2 = \begin{pmatrix} 3 & -6 & 3 \\ 3 & -6 & 3 \\ 3 & -6 & 3 \end{pmatrix}, \\
\mathcal{M}O_0 &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ \mathcal{M}O_1 = \begin{pmatrix} 6 & -6 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \ \mathcal{M}O_2 = \begin{pmatrix} 12 & -16 & 6 \\ 6 & -6 & 2 \\ 2 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}p_0 &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & -6 & 7 \end{pmatrix}, \ \mathcal{M}p_1 = \begin{pmatrix} 5 & -3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \ \mathcal{M}p_2 = \begin{pmatrix} 12 & -14 & 5 \\ 5 & -3 & 1 \\ 1 & 2 & 0 \end{pmatrix}.
\end{aligned}$$

The sequences $(\mathcal{M}T_n)$, $(\mathcal{M}H_n)$, $(\mathcal{M}O_n)$ and $(\mathcal{M}p_n)$ can be extended to negative subscripts by defining

$$\begin{aligned}
\mathcal{M}T_{-n} &= 3\mathcal{M}T_{-(n-1)} - 3\mathcal{M}T_{-(n-2)} + \mathcal{M}T_{-(n-3)}, \\
\mathcal{M}H_{-n} &= 3\mathcal{M}H_{-(n-1)} - 3\mathcal{M}H_{-(n-2)} + \mathcal{M}H_{-(n-3)}, \\
\mathcal{M}O_{-n} &= 3\mathcal{M}O_{-(n-1)} - 3\mathcal{M}O_{-(n-2)} + \mathcal{M}O_{-(n-3)}, \\
\mathcal{M}p_{-n} &= 3\mathcal{M}p_{-(n-1)} - 3\mathcal{M}p_{-(n-2)} + \mathcal{M}p_{-(n-3)}
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

The following theorem gives the nth general terms of the generalized Guglielmo matrix sequence.

Theorem 2.1. *For any integer n, we have the following formulas of the generalized Guglielmo matrix sequence:*

$$\mathcal{M}W_n = \begin{pmatrix} W_{n+1} & -3W_n + W_{n-1} & W_n \\ W_n & -3W_{n-1} + W_{n-2} & W_{n-1} \\ W_{n-1} & -3W_{n-2} + W_{n-3} & W_{n-2} \end{pmatrix}. \tag{2.2}$$

Proof. Suppose that $n \geq 0$. We prove (2.2) by mathematical induction on n. If $n = 0$, since $W_{-1} = 3W_0 - 3W_1 + W_2$, $W_{-2} = 6W_0 - 8W_1 + 3W_2$, $W_{-3} = 10W_0 - 15W_1 + 6W_2$, we have

$$\begin{aligned}
\mathcal{M}W_0 &= \begin{pmatrix} W_1 & -3W_0 + W_{-1} & W_0 \\ W_0 & -3W_{-1} + W_{-2} & W_{-1} \\ W_{-1} & -3W_{-2} + W_{-3} & W_{-2} \end{pmatrix} \\
&= \begin{pmatrix} W_1 & W_2 - 3W_1 & W_0 \\ W_0 & W_1 - 3W_0 & 3W_0 - 3W_1 + W_2 \\ 3W_0 - 3W_1 + W_2 & 9W_1 - 8W_0 - 3W_2 & 6W_0 - 8W_1 + 3W_2 \end{pmatrix}
\end{aligned}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k + 1$, we have

$$\begin{aligned}
\mathcal{M}W_{k+1} &= 3\mathcal{M}W_k - 3\mathcal{M}W_{k-1} + \mathcal{M}W_{k-2} \\
&= 3\begin{pmatrix} W_{k+1} & -3W_k + W_{k-1} & W_k \\ W_k & -3W_{k-1} + W_{k-2} & W_{k-1} \\ W_{k-1} & -3W_{k-2} + W_{k-3} & W_{k-2} \end{pmatrix} - 3\begin{pmatrix} W_{(k-1)+1} & -3W_{(k-1)} + W_{(k-1)-1} & W_{(k-1)} \\ W_{(k-1)} & -3W_{(k-1)-1} + W_{(k-1)-2} & W_{(k-1)-1} \\ W_{(k-1)-1} & -3W_{(k-1)-2} + W_{(k-1)-3} & W_{(k-1)-2} \end{pmatrix} \\
&\quad + \begin{pmatrix} W_{(k-2)+1} & -3W_{(k-2)} + W_{(k-2)-1} & W_{(k-2)} \\ W_{(k-2)} & -3W_{(k-2)-1} + W_{(k-2)-2} & W_{(k-2)-1} \\ W_{(k-2)-1} & -3W_{(k-2)-2} + W_{(k-2)-3} & W_{(k-2)-2} \end{pmatrix} \\
&= \begin{pmatrix} W_{(k+1)+1} & -3W_{k+1} + W_{(k+1)-1} & W_{k+1} \\ W_{k+1} & -3W_{(k+1)-1} + W_{(k+1)-2} & W_{(k+1)-1} \\ W_{(k+1)-1} & -3W_{(k+1)-2} + W_{(k+1)-3} & W_{(k+1)-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k + 1$, this proves (2.2).

For the case $n \leq 0$, (2.2) can be proved by strong mathematical induction on n. □

The following corollary gives the nth general terms of the the triangular matrix $(\mathcal{M}T_n)$, Lucas-triangular matrix $(\mathcal{M}H_n)$, oblong matrix $(\mathcal{M}O_n)$ and pentegonal matrix $(\mathcal{M}p_n)$.

Corollary 2.2. *For any integer n, we have the following formulas of the matrix sequences:*

$$\begin{aligned}
\mathcal{M}T_n &= \begin{pmatrix} T_{n+1} & -3T_n+T_{n-1} & T_n \\ T_n & -3T_{n-1}+T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2}+T_{n-3} & T_{n-2} \end{pmatrix}, \\
\mathcal{M}H_n &= \begin{pmatrix} H_{n+1} & -3H_n+H_{n-1} & H_n \\ H_n & -3H_{n-1}+H_{n-2} & H_{n-1} \\ H_{n-1} & -3H_{n-2}+H_{n-3} & H_{n-2} \end{pmatrix}, \\
\mathcal{M}O_n &= \begin{pmatrix} O_{n+1} & -3O_n+O_{n-1} & O_n \\ O_n & -3O_{n-1}+O_{n-2} & O_{n-1} \\ O_{n-1} & -3O_{n-2}+O_{n-3} & O_{n-2} \end{pmatrix}, \\
\mathcal{M}p_n &= \begin{pmatrix} p_{n+1} & -3p_n+p_{n-1} & p_n \\ p_n & -3p_{n-1}+p_{n-2} & p_{n-1} \\ p_{n-1} & -3p_{n-2}+p_{n-3} & p_{n-2} \end{pmatrix}.
\end{aligned}$$

Lemma 2.3. *We suppose that $\alpha = \beta = \gamma$, the Binet's formula of generalized (r, s, t) sequence is*

$$W_n = (A_1 + A_2 n + A_3 n^2) \times \alpha^n \tag{2.3}$$

where α, β, γ are the roots of the characteristic equation of the generalized (r, s, t) sequence and

$$\begin{aligned}
A_1 &= W_0, \\
A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\
A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0).
\end{aligned}$$

We now give the Binet's formula for the generalized Guglielmo matrix sequence.

Theorem 2.4. *For every integer n, the Binet's formula of the generalized Guglielmo matrix sequence are given by*

$$\mathcal{M}W_n = \mathcal{M}A_1 + \mathcal{M}A_2 n + \mathcal{M}A_3 n^2 \tag{2.4}$$

where

$$\begin{aligned}
\mathcal{M}A_1 &= \mathcal{M}W_0, \\
\mathcal{M}A_2 &= \frac{1}{2}(-\mathcal{M}W_2 + 4\mathcal{M}W_1 - 3\mathcal{M}W_0), \\
\mathcal{M}A_3 &= \frac{1}{2}(\mathcal{M}W_2 - 2\mathcal{M}W_1 + \mathcal{M}W_0).
\end{aligned}$$

Proof. The proof can be done easily by using Lemma 2.3. □

The following corollory gives the Binet's formulas of the triangular, Lucas-triangular, oblong and pentagonals matrix sequences.

Corollary 2.5. *For every integer n, the Binet's formulas of the triangular, Lucas-triangular, oblong and pentagonals matrix sequences, respectively, are given by*

(a)

$$\mathcal{M}T_n = \mathcal{M}B_1 + \mathcal{M}B_2 n + \mathcal{M}B_3 n^2$$

where

$$\begin{aligned}
\mathcal{M}B_1 &= \mathcal{M}T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathcal{M}B_2 &= \frac{1}{2}(-\mathcal{M}T_2 + 4\mathcal{M}T_1 - 3\mathcal{M}T_0) = \begin{pmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{pmatrix}, \\
\mathcal{M}B_3 &= \frac{1}{2}(\mathcal{M}T_2 - 2\mathcal{M}T_1 + \mathcal{M}T_0) = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

i.e.

$$\mathcal{M}T_n = \begin{pmatrix} \frac{1}{2}n^2 + \frac{3}{2}n + 1 & -n^2 - 2n & \frac{1}{2}n^2 + \frac{1}{2}n \\ \frac{1}{2}n^2 + \frac{1}{2}n & 1 - n^2 & \frac{1}{2}n^2 - \frac{1}{2}n \\ \frac{1}{2}n^2 - \frac{1}{2}n & 2n - n^2 & \frac{1}{2}n^2 - \frac{3}{2}n + 1 \end{pmatrix}.$$

(b)

$$\mathcal{M}H_n = \mathcal{M}C_1 + \mathcal{M}C_2 n + \mathcal{M}C_3 n^2$$

where

$$\begin{aligned}
\mathcal{M}C_1 &= \mathcal{M}H_0 = \begin{pmatrix} 3 & -6 & 3 \\ 3 & -6 & 3 \\ 3 & -6 & 3 \end{pmatrix}, \\
\mathcal{M}C_2 &= \frac{1}{2}(-\mathcal{M}H_2 + 4\mathcal{M}H_1 - 3\mathcal{M}H_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}C_3 &= \frac{1}{2}(\mathcal{M}H_2 - 2\mathcal{M}H_1 + \mathcal{M}H_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

i.e.

$$\mathcal{M}H_n = \begin{pmatrix} 3 & -6 & 3 \\ 3 & -6 & 3 \\ 3 & -6 & 3 \end{pmatrix}.$$

(c)

$$\mathcal{M}O_n = \mathcal{M}D_1 + \mathcal{M}D_2 n + \mathcal{M}D_3 n^2$$

where

$$\begin{aligned}
\mathcal{M}D_1 &= \mathcal{M}O_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \\
\mathcal{M}D_2 &= \frac{1}{2}(-\mathcal{M}O_2 + 4\mathcal{M}O_1 - 3\mathcal{M}O_0) = \begin{pmatrix} 3 & -4 & 1 \\ 1 & 0 & -1 \\ -1 & 4 & -3 \end{pmatrix}, \\
\mathcal{M}D_3 &= \frac{1}{2}(\mathcal{M}O_2 - 2\mathcal{M}O_1 + \mathcal{M}O_0) = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}.
\end{aligned}$$

i.e.

$$\mathcal{M}O_n = \begin{pmatrix} n^2 + 3n + 2 & -2n^2 - 4n & n^2 + n \\ n^2 + n & 2 - 2n^2 & n^2 - n \\ n^2 - n & 4n - 2n^2 & n^2 - 3n + 2 \end{pmatrix}.$$

(d)

$$\mathcal{M}p_n = \mathcal{M}E_1 + \mathcal{M}E_2 n + \mathcal{M}E_3 n^2$$

where

$$\begin{aligned} \mathcal{M}E_1 &= \mathcal{M}p_0 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & -6 & 7 \end{pmatrix}, \\ \mathcal{M}E_2 &= \frac{1}{2}(-\mathcal{M}p_2 + 4\mathcal{M}p_1 - 3\mathcal{M}p_0) = \begin{pmatrix} \frac{5}{2} & -2 & -\frac{1}{2} \\ -\frac{1}{2} & 4 & -\frac{7}{2} \\ -\frac{7}{2} & 10 & -\frac{13}{2} \end{pmatrix}, \\ \mathcal{M}E_3 &= \frac{1}{2}(\mathcal{M}p_2 - 2\mathcal{M}p_1 + \mathcal{M}p_0) = \begin{pmatrix} \frac{3}{2} & -3 & \frac{3}{2} \\ \frac{3}{2} & -3 & \frac{3}{2} \\ \frac{3}{2} & -3 & \frac{3}{2} \end{pmatrix}. \end{aligned}$$

i.e.

$$\mathcal{M}p_n = \begin{pmatrix} \frac{3}{2}n^2 + \frac{5}{2}n + 1 & -3n^2 - 2n + 2 & \frac{3}{2}n^2 - \frac{1}{2}n \\ \frac{3}{2}n^2 - \frac{1}{2}n & -3n^2 + 4n + 1 & \frac{3}{2}n^2 - \frac{7}{2}n + 2 \\ \frac{3}{2}n^2 - \frac{7}{2}n + 2 & -3n^2 + 10n - 6 & \frac{3}{2}n^2 - \frac{13}{2}n + 7 \end{pmatrix}.$$

Now, we present some summation formulas for the generalized Guglielmo matrix sequence.

Theorem 2.6. *For all integers m, j, we have the following formulas.*

(a)

$$\begin{aligned} \sum_{k=0}^{n-1} \mathcal{M}W_{km+j} &= \sum_{k=0}^{n-1} (n\mathcal{M}A_1 + \mathcal{M}A_2 \frac{n(2j - m + mn)}{2} && (2.5) \\ &\quad + \mathcal{M}A_3 \frac{n(6j^2 + 6jmn - 6jm + 2m^2n^2 - 3m^2n + m^2)}{6}). && (2.6) \end{aligned}$$

(b)

$$\begin{aligned} \sum_{k=0}^{n-1} k\mathcal{M}W_{km+j} &= \mathcal{M}A_1 \frac{n(n-1)}{2} + \mathcal{M}A_2 \frac{n(n-1)(3j - m + 2mn)}{6} \\ &\quad + \mathcal{M}A_3 \frac{n(n-1)(6j^2 + 8jmn - 4jm + 3m^2n^2 - 3m^2n)}{12}. \end{aligned}$$

Proof.

(a) Using (2.4), we get

$$\begin{aligned} \sum_{k=0}^{n-1} \mathcal{M}W_{km+j} &= \sum_{k=0}^{n-1} (\mathcal{M}A_1 + \mathcal{M}A_2 (j + km) + \mathcal{M}A_3 (j + km)^2) && (2.7) \\ &= \sum_{k=0}^{n-1} \mathcal{M}A_1 + \sum_{k=0}^{n-1} \mathcal{M}A_2 (j + km) + \sum_{k=0}^{n-1} \mathcal{M}A_3 (j + km)^2 \\ &= \sum_{k=0}^{n-1} (n\mathcal{M}A_1 + \mathcal{M}A_2 \frac{n(2j - m + mn)}{2} \\ &\quad + \mathcal{M}A_3 \frac{n\left(6j^2 + 6jmn - 6jm + 2m^2n^2 - 3m^2n + m^2\right)}{6}). && (2.8) \end{aligned}$$

(b) Using (2.4), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} k\mathcal{M}W_{km+j} &= \sum_{k=0}^{n-1}(k\mathcal{M}A_1 + \mathcal{M}A_2 k\,(j+km) + \mathcal{M}A_3 k\,(j+km)^2 \\
&= \sum_{k=0}^{n-1} k\mathcal{M}A_1 + \sum_{k=0}^{n-1}\mathcal{M}A_2 k\,(j+km) + \sum_{k=0}^{n-1}\mathcal{M}A_3 k\,(j+km)^2 \\
&= \mathcal{M}A_1 \frac{n\,(n-1)}{2} + \mathcal{M}A_2 \frac{n\,(n-1)\,(3j-m+2mn)}{6} \\
&\quad + \mathcal{M}A_3 \frac{n\,(n-1)\,(6j^2+8jmn-4jm+3m^2n^2-3m^2n)}{12}. \;\square
\end{aligned}$$

From the theorem given above, we obtain the following corollary, which deals with the summation formulas for the triangular, Lucas-triangular, oblong, and pentagonal matrix sequences.

Corollary 2.7. *For all integers m, j we have the following formulas.*

(a)

$$\begin{aligned}
\sum_{k=0}^{n-1} \mathcal{M}T_{km+j} &= \sum_{k=0}^{n-1}(n\mathcal{M}B_1 + \mathcal{M}B_2 \frac{n(2j-m+mn)}{2} \\
&\quad + \mathcal{M}B_3 \frac{n(6j^2+6jmn-6jm+2m^2n^2-3m^2n+m^2)}{6}), \\
\sum_{k=0}^{n-1} k\mathcal{M}T_{km+j} &= \mathcal{M}B_1 \frac{n\,(n-1)}{2} + \mathcal{M}B_2 \frac{n\,(n-1)\,(3j-m+2mn)}{6} \\
&\quad + \mathcal{M}B_3 \frac{n\,(n-1)\,(6j^2+8jmn-4jm+3m^2n^2-3m^2n)}{12}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}B_1 &= \mathcal{M}T_0, \\
\mathcal{M}B_2 &= \frac{1}{2}(-\mathcal{M}T_2 + 4\mathcal{M}T_1 - 3\mathcal{M}T_0), \\
\mathcal{M}B_3 &= \frac{1}{2}(\mathcal{M}T_2 - 2\mathcal{M}T_1 + \mathcal{M}T_0).
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{k=0}^{n-1} \mathcal{M}O_{km+j} &= \sum_{k=0}^{n-1}(n\mathcal{M}C_1 + \mathcal{M}C_2 \frac{n(2j-m+mn)}{2} \\
&\quad + \mathcal{M}C_3 \frac{n(6j^2+6jmn-6jm+2m^2n^2-3m^2n+m^2)}{6}), \\
\sum_{k=0}^{n-1} k\mathcal{M}O_{km+j} &= \mathcal{M}C_1 \frac{n\,(n-1)}{2} + \mathcal{M}C_2 \frac{n\,(n-1)\,(3j-m+2mn)}{6} \\
&\quad + \mathcal{M}C_3 \frac{n\,(n-1)\,(6j^2+8jmn-4jm+3m^2n^2-3m^2n)}{12}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}C_1 &= \mathcal{M}H_0,\\
\mathcal{M}C_2 &= \frac{1}{2}(-\mathcal{M}H_2+4\mathcal{M}H_1-3\mathcal{M}H_0),\\
\mathcal{M}C_3 &= \frac{1}{2}(\mathcal{M}H_2-2\mathcal{M}H_1+\mathcal{M}H_0).
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{k=0}^{n-1}\mathcal{M}O_{km+j} &= \sum_{k=0}^{n-1}(n\mathcal{M}D_1+\mathcal{M}D_2\frac{n(2j-m+mn)}{2}\\
&\quad +\mathcal{M}D_3\frac{n(6j^2+6jmn-6jm+2m^2n^2-3m^2n+m^2)}{6}),\\
\sum_{k=0}^{n-1}k\mathcal{M}O_{km+j} &= \mathcal{M}D_1\frac{n(n-1)}{2}+\mathcal{M}D_2\frac{n(n-1)(3j-m+2mn)}{6}\\
&\quad +\mathcal{M}D_3\frac{n(n-1)(6j^2+8jmn-4jm+3m^2n^2-3m^2n)}{12}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}D_1 &= \mathcal{M}O_0,\\
\mathcal{M}D_2 &= \frac{1}{2}(-\mathcal{M}O_2+4\mathcal{M}O_1-3\mathcal{M}O_0),\\
\mathcal{M}D_3 &= \frac{1}{2}(\mathcal{M}O_2-2\mathcal{M}O_1+\mathcal{M}O_0).
\end{aligned}$$

(d)

$$\begin{aligned}
\sum_{k=0}^{n-1}\mathcal{M}p_{km+j} &= \sum_{k=0}^{n-1}(n\mathcal{M}E_1+\mathcal{M}E_2\frac{n(2j-m+mn)}{2}\\
&\quad +\mathcal{M}E_3\frac{n(6j^2+6jmn-6jm+2m^2n^2-3m^2n+m^2)}{6}),\\
\sum_{k=0}^{n-1}k\mathcal{M}p_{km+j} &= \mathcal{M}E_1\frac{n(n-1)}{2}+\mathcal{M}E_2\frac{n(n-1)(3j-m+2mn)}{6}\\
&\quad +\mathcal{M}E_3\frac{n(n-1)(6j^2+8jmn-4jm+3m^2n^2-3m^2n)}{12}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}E_1 &= \mathcal{M}p_0,\\
\mathcal{M}E_2 &= \frac{1}{2}(-\mathcal{M}p_2+4\mathcal{M}p_1-3\mathcal{M}p_0),\\
\mathcal{M}E_3 &= \frac{1}{2}(\mathcal{M}p_2-2\mathcal{M}p_1+\mathcal{M}p_0).
\end{aligned}$$

Next, we present the generating function of the generalized Guglielmo matrix sequence.

Theorem 2.8. *Let $f_{\mathcal{M}W_n}(x)=\sum_{n=0}^{\infty}\mathcal{M}W_n x^n$ and $A=(a_{ij})_{3\times 3}$ denote the generating function of generalized Guglielmo matrix sequences. Then,*

$$\begin{aligned} f_{\mathcal{M}W_n}(x) &= \sum_{n=0}^{\infty}\mathcal{M}W_n x^n = \frac{\mathcal{M}W_0+(\mathcal{M}W_1-3\mathcal{M}W_0)x+(\mathcal{M}W_2-3\mathcal{M}W_1+3\mathcal{M}W_0)x^2}{1-3x+3x^2-x^3} \\ &= \frac{1}{1-3x+3x^2-x^3}A. \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} a_{11} &= W_1+(-3W_1+W_2)x+W_0x^2, \\ a_{21} &= W_0+(-3W_0+W_1)x+3x^2W_0+(-3W_1+W_2)x^2, \\ a_{31} &= (3W_0-3W_1+W_2)+(-8W_0+9W_1-3W_2)x+(6W_0-8W_1+3W_2)x^2, \end{aligned}$$

and

$$\begin{aligned} a_{12} &= W_2-3W_1+(W_0+6W_1-3W_2)x+(-3W_0+W_1)x^2 \\ a_{22} &= W_1-3W_0+(9W_0-6W_1+W_2)x+(-8W_0+9W_1-3W_2)x^2 \\ a_{32} &= 9W_1-8W_0-3W_2+(21W_0-26W_1+9W_2)x+(-15W_0+21W_1-8W_2)x^2 \end{aligned}$$

and

$$\begin{aligned} a_{13} &= W_0+(-3W_0+W_1)x+(3W_0-3W_1+W_2)x^2, \\ a_{23} &= (3W_0-3W_1+W_2)+(-8W_0+9W_1-3W_2)x+(6W_0-8W_1+3W_2)x^2, \\ a_{33} &= (6W_0-8W_1+3W_2)+(-15W_0+21W_1-8W_2)x+(10W_0-15W_1+6W_2)x^2. \end{aligned}$$

Proof. Using the definition of generalized Guglielmo matrix sequences, and substracting $xf_{\mathcal{M}W_n}(x)$, $x^2f_{\mathcal{M}W_n}(x)$ and $x^3f_{\mathcal{M}W_n}(x)$ from $f_{\mathcal{M}W_n}(x)$ we get

$$\begin{aligned} (1-3x+3x^2-x^3)f_{\mathcal{M}W_n}(x) &= \sum_{n=0}^{\infty}\mathcal{M}W_n x^n-3x\sum_{n=0}^{\infty}\mathcal{M}W_n x^n+3x^2\sum_{n=0}^{\infty}\mathcal{M}W_n x^n-x^3\sum_{n=0}^{\infty}\mathcal{M}W_n x^n \\ &= \sum_{n=0}^{\infty}\mathcal{M}W_n x^n-3\sum_{n=0}^{\infty}\mathcal{M}W_n x^{n+1}+3\sum_{n=0}^{\infty}\mathcal{M}W_n x^{n+2}-\sum_{n=0}^{\infty}\mathcal{M}W_n x^{n+3} \\ &= \sum_{n=0}^{\infty}\mathcal{M}W_n x^n-3\sum_{n=1}^{\infty}\mathcal{M}W_{n-1} x^{n}+3\sum_{n=2}^{\infty}\mathcal{M}W_{n-2} x^{n}-\sum_{n=3}^{\infty}\mathcal{M}W_{n-3} x^{n} \\ &= (\mathcal{M}W_0+\mathcal{M}W_1x+\mathcal{M}W_2x^2)-3(\mathcal{M}W_0x+\mathcal{M}W_1x^2)+3\mathcal{M}W_0x^2 \\ &\quad +\sum_{n=3}^{\infty}(\mathcal{M}W_n-3\mathcal{M}W_{n-1}+3\mathcal{M}W_{n-2}-\mathcal{M}W_{n-3})x^n \\ &= \mathcal{M}W_0+\mathcal{M}W_1x+\mathcal{M}W_2x^2-3\mathcal{M}W_0x-3\mathcal{M}W_1x^2+3\mathcal{M}W_0x^2 \\ &= \mathcal{M}W_0+(\mathcal{M}W_1-3\mathcal{M}W_0)x+(\mathcal{M}W_2-3\mathcal{M}W_1+3\mathcal{M}W_0)x^2. \end{aligned}$$

Rearranging the above equation, we obtain

$$\sum_{n=0}^{\infty}\mathcal{M}W_n x^n=\frac{\mathcal{M}W_0+(\mathcal{M}W_1-3\mathcal{M}W_0)x+(\mathcal{M}W_2-3\mathcal{M}W_1+3\mathcal{M}W_0)x^2}{1-3x+3x^2-x^3}$$

which equals the identity that stated in the in the Theorem. This completes the proof. □

The following corollary gives the generating functions of the triangular, Lucas-triangular, oblong and pentagonal matrix sequences.

Corollary 2.9. *The generating functions for the triangular, Lucas-triangular, oblong and pentagonal matrix sequences, respectively, are given as*

(a) *If we take $\mathcal{M}W_n = \mathcal{M}T_n$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty} \mathcal{M}W_n x^n = \frac{1}{1-3x+3x^2-x^3} \begin{pmatrix} 1 & x(x-3) & x \\ x & 1-3x & x^2 \\ x^2 & x-3x^2 & 3x^2-3x+1 \end{pmatrix}.$$

(b) *If we take $\mathcal{M}W_n = \mathcal{M}H_n$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty} \mathcal{M}W_n x^n = \frac{1}{1-3x+3x^2-x^3} \begin{pmatrix} 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \end{pmatrix}.$$

(c) *If we take $\mathcal{M}W_n = \mathcal{M}O_n$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty} \mathcal{M}W_n x^n = \frac{1}{1-3x+3x^2-x^3} \begin{pmatrix} 2 & 2x(x-3) & 2x \\ 2x & 2-6x & 2x^2 \\ 2x^2 & -2x(3x-1) & 6x^2-6x+2 \end{pmatrix}.$$

(d) *If we take $\mathcal{M}W_n = \mathcal{M}p_n$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty} \mathcal{M}W_n x^n = \frac{1}{1-3x+3x^2-x^3} \begin{pmatrix} 2x+1 & x^2-9x+2 & x(2x+1) \\ x(2x+1) & -6x^2-x+1 & 7x^2-6x+2 \\ 7x^2-6x+2 & -19x^2+19x-6 & 15x^2-19x+7 \end{pmatrix}.$$

Next, we give the generating function of the generalized Guglielmo matrix sequences with negative indices.

Theorem 2.10. *Let $g_{\mathcal{M}W_n}(x) = \sum_{n=0}^{\infty} \mathcal{M}W_{-n} x^n$ be generating function of the generalized Guglielmo matrix sequence. For negative indices, the generating function for the generalized Guglielmo matrix sequence is given as follows*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{M}W_n x^n &= \frac{\mathcal{M}W_0 + (\mathcal{M}W_{-1} - 3\mathcal{M}W_0)x + (\mathcal{M}W_{-2} - 3\mathcal{M}W_{-1} + 3\mathcal{M}W_0)x^2}{1-3x+3x^2-x^3} \qquad (2.10) \\ &= \frac{1}{1-3x+3x^2-x^3} B \end{aligned}$$

where $B = (b_{ij})_{3\times 3}$ and

$$\begin{aligned} b_{11} &= W_1 + (W_0 - 3W_1)x + x^2 W_2, \\ b_{21} &= W_0 + (-3W_1 + W_2)x + x^2 W_1, \\ b_{31} &= (3W_0 - 3W_1 + W_2) + (-3W_0 + W_1)x + x^2 W_0 \end{aligned}$$

and

$$\begin{aligned} b_{12} &= W_2 - 3W_1 + (-3W_0 + 10W_1 - 3W_2)x + (W_0 - 3W_1)x^2, \\ b_{22} &= W_1 - 3W_0 + (W_0 + 6W_1 - 3W_2)x + (-3W_1 + W_2)x^2, \\ b_{32} &= 9W_1 - 8W_0 + -3W_2 + (9W_0 - 6W_1 + W_2)x + (-3W_0 + W_1)x^2 \end{aligned}$$

and

$$\begin{aligned} b_{13} &= W_0 + (-3W_1 + W_2)x + x^2 W_1, \\ b_{23} &= (3W_0 - 3W_1 + W_2) + (-3W_0 + W_1)x + x^2 W_0, \\ b_{33} &= (6W_0 - 8W_1 + 3W_2) + (-8W_0 + 9W_1 - 3W_2)x + (3W_0 - 3W_1 + W_2)x^2. \end{aligned}$$

Proof. Using the definition of generalized Guglielmo matrix sequences, and substracting $xg_{\mathcal{M}W_n}(x)$, $x^2g_{\mathcal{M}W_n}(x)$ and $x^3g_{\mathcal{M}W_n}(x)$ from $g_{\mathcal{M}W_n}(x)$ we get

$$\begin{aligned}
(1-3x+3x^2-x^3)f_{\mathcal{M}W_{-n}}(x) &= \sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n - 3x\sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n + 3x^2\sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n - x^3\sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n \\
&= \sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n - 3\sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^{n+1} + 3\sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^{n+2} - \sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^{n+3} \\
&= \sum_{n=0}^{\infty}\mathcal{M}W_{-n}x^n - 3\sum_{n=1}^{\infty}\mathcal{M}W_{-(n-1)}x^n + 3\sum_{n=2}^{\infty}\mathcal{M}W_{-(n-2)}x^n - \sum_{n=3}^{\infty}\mathcal{M}W_{-(n-3)}x^n \\
&= (\mathcal{M}W_0 + \mathcal{M}W_{-1}x + \mathcal{M}W_{-2}x^2) - 3(\mathcal{M}W_0x + \mathcal{M}W_{-1}x^2) + 3\mathcal{M}W_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(3\mathcal{M}W_{-(n-1)} - 3\mathcal{M}W_{-(n-2)} + \mathcal{M}W_{-(n-3)})x^n \\
&= \mathcal{M}W_0 + \mathcal{M}W_{-1}x + \mathcal{M}W_{-2}x^2 - 3\mathcal{M}W_0x - 3\mathcal{M}W_{-1}x^2 + 3\mathcal{M}W_0x^2 \\
&= \mathcal{M}W_0 + (\mathcal{M}W_{-1} - 3\mathcal{M}W_0)x + (\mathcal{M}W_{-2} - 3\mathcal{M}W_{-1} + 3\mathcal{M}W_0)x^2.
\end{aligned}$$

Rearranging above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{M}W_nx^n = \frac{\mathcal{M}W_0 + (\mathcal{M}W_{-1} - 3\mathcal{M}W_0)x + (\mathcal{M}W_{-2} - 3\mathcal{M}W_{-1} + 3\mathcal{M}W_0)x^2}{1-3x+3x^2-x^3}$$

which equals the identity that stated in the Theorem. □

The following corollary gives the generating functions of the triangular, Lucas-triangular, oblong and pentagonal matrix sequences with negative subscript .

Corollary 2.11. *The generating functions for the triangular, Lucas-triangular, oblong and pentagonal matrix sequences with negative subscript, respectively, are given as*

(a) *If we take $\mathcal{M}W_{-n} = \mathcal{M}T_{-n}$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty}\mathcal{M}T_{-n}x^n = \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} 3x^2-3x+1 & x-3x^2 & x^2 \\ x^2 & 1-3x & x \\ x & x(x-3) & 1 \end{pmatrix}.$$

(b) *If we take $\mathcal{M}W_{-n} = \mathcal{M}H_{-n}$ in the Theorem 2.8, we have,*

$$\sum_{n=0}^{\infty}\mathcal{M}H_{-n}x^n = \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \end{pmatrix}.$$

(c) *If we take $\mathcal{M}W_{-n} = \mathcal{M}O_{-n}$ in the Theorem 2.8, we obtain,*

$$\sum_{n=0}^{\infty}\mathcal{M}O_{-n}x^n = \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} 6x^2-6x+2 & -2x(3x-1) & 2x^2 \\ 2x^2 & 2-6x & 2x \\ 2x & 2x(x-3) & 2 \end{pmatrix}.$$

(d) *If we take $\mathcal{M}W_{-n} = \mathcal{M}p_{-n}$ in the Theorem 2.8, we get,*

$$\sum_{n=0}^{\infty}\mathcal{M}p_{-n}x^n = \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} 5x^2-3x+1 & -3x^2-5x+2 & x(x+2) \\ x(x+2) & 2x^2-9x+1 & x+2 \\ x+2 & x^2-x-6 & 2x^2-6x+7 \end{pmatrix}.$$

Now, we give two special equality in the following lemma that we need rest of the study.

Lemma 2.12. *([Frontczak, (2018)])We assume that* $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$*. Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x)=\sum_{n=0}^{\infty} a_{2n}x^n=\frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x)=\sum_{n=0}^{\infty} a_{2n+1}x^n=\frac{f(\sqrt{x})-f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

Theorem 2.13. *The generating functions of the sequence* $\mathcal{M}W_{2n}$ *and* $\mathcal{M}W_{2n+1}$ *are provided by*

$$f_{\mathcal{M}W_{2n}}(x)=\frac{\mathcal{M}W_0+(\mathcal{M}W_2-3\mathcal{M}W_0)x+(6\mathcal{M}W_0-8\mathcal{M}W_1+3\mathcal{M}W_2)x^2}{1-3x+3x^2-x^3}, \quad (2.11)$$

$$f_{\mathcal{M}W_{2n+1}}(x)=\frac{\mathcal{M}W_1+(\mathcal{M}W_0-6\mathcal{M}W_1+3\mathcal{M}W_2)x+(3\mathcal{M}W_0-3\mathcal{M}W_1+\mathcal{M}W_2)x^2}{1-3x+3x^2-x^3}. \quad (2.12)$$

Proof. We only proof (2.11). From Theorem (2.8) we can obtain following identities:

$$\begin{aligned} f_{\mathcal{M}W_n}(\sqrt{x}) &= \frac{\mathcal{M}W_0-\sqrt{x}\,(\mathcal{M}W_1-3\mathcal{M}W_0)+x\,(3\mathcal{M}W_0-3\mathcal{M}W_1+\mathcal{M}W_2)}{3x+3\sqrt{x}+x^{\frac{3}{2}}+1}. \\ f_{\mathcal{M}W_n}(-\sqrt{x}) &= -\frac{\mathcal{M}W_0+\sqrt{x}\,(\mathcal{M}W_1-3\mathcal{M}W_0)+x\,(3\mathcal{M}W_0-3\mathcal{M}W_1+\mathcal{M}W_2)}{3\sqrt{x}-3x+x^{\frac{3}{2}}-1}. \end{aligned}$$

Thereby, using lemma (2.12) identity (2.11) can be proved . The other identity can be found similarly. □
In the next corollary, we present the generating function of generalized Guglielmo matrix sequences with odd and even subscript by using the Theorem 2.13

Corollary 2.14. *The triangular, Lucas-triangular, oblong, and pentagonal matrix sequences with even and odd subscript, respectively, are given as*

(a) *If we take* $\mathcal{M}W_n=\mathcal{M}T_n$ *in the Theorem 2.13, we get,*

$$\begin{aligned} f_{\mathcal{M}T_{2n}}(x) &= \frac{\mathcal{M}T_0+(\mathcal{M}T_2-3\mathcal{M}T_0)x+(6\mathcal{M}T_0-8\mathcal{M}T_1+3\mathcal{M}T_2)x^2}{1-3x+3x^2-x^3} \\ &= \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} 3x+1 & -8x & x\,(x+3) \\ x\,(x+3) & -3x^2-6x+1 & x\,(3x+1) \\ x\,(3x+1) & -8x^2 & 6x^2-3x+1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} f_{\mathcal{M}T_{2n+1}}(x) &= \frac{\mathcal{M}T_1+(\mathcal{M}T_0-6\mathcal{M}T_1+3\mathcal{M}T_2)x+(3\mathcal{M}T_0-3\mathcal{M}T_1+\mathcal{M}T_2)x^2}{1-3x+3x^2-x^3} \\ &= \frac{1}{1-3x+3x^2-x^3}\begin{pmatrix} x+3 & x^2-6x-3 & 3x+1 \\ 3x+1 & -8x & x\,(x+3) \\ x\,(x+3) & -3x^2-6x+1 & x\,(3x+1) \end{pmatrix}. \end{aligned}$$

(b) *If we take $\mathcal{M}W_n = \mathcal{M}H_n$ in the Theorem 2.13, we get,*

$$\begin{aligned} f_{\mathcal{M}H_{2n}}(x) &= \frac{\mathcal{M}H_0 + (\mathcal{M}H_2 - 3\mathcal{M}H_0)x + (6\mathcal{M}H_0 - 8\mathcal{M}H_1 + 3\mathcal{M}H_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} f_{\mathcal{M}H_{2n+1}}(x) &= \frac{\mathcal{M}H_1 + (\mathcal{M}H_0 - 6\mathcal{M}H_1 + 3\mathcal{M}H_2)x + (3\mathcal{M}H_0 - 3\mathcal{M}H_1 + \mathcal{M}H_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \\ 3(x-1)^2 & -6(x-1)^2 & 3(x-1)^2 \end{pmatrix}. \end{aligned}$$

(c) *If we take $\mathcal{M}W_n = \mathcal{M}O_n$ in the Theorem 2.13, we get,*

$$\begin{aligned} f_{\mathcal{M}O_{2n}}(x) &= \frac{\mathcal{M}O_0 + (\mathcal{M}O_2 - 3\mathcal{M}O_0)x + (6\mathcal{M}O_0 - 8\mathcal{M}O_1 + 3\mathcal{M}O_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 6x+2 & -16x & 2x(x+3) \\ 2x(x+3) & -6x^2 - 12x + 2 & 2x(3x+1) \\ 2x(3x+1) & -16x^2 & 12x^2 - 6x + 2 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} f_{\mathcal{M}O_{2n+1}}(x) &= \frac{\mathcal{M}O_1 + (\mathcal{M}O_0 - 6\mathcal{M}O_1 + 3\mathcal{M}O_2)x + (3\mathcal{M}O_0 - 3\mathcal{M}O_1 + \mathcal{M}O_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 2x+6 & 2x^2 - 12x - 6 & 6x+2 \\ 6x+2 & -16x & 2x(x+3) \\ 2x(x+3) & -6x^2 - 12x + 2 & 2x(3x+1) \end{pmatrix}. \end{aligned}$$

(d) *If we take $\mathcal{M}O_n = \mathcal{M}p_n$ in the Theorem 2.13, we get,*

$$\begin{aligned} f_{\mathcal{M}p_{2n}}(x) &= \frac{\mathcal{M}p_0 + (\mathcal{M}p_2 - 3\mathcal{M}p_0)x + (6\mathcal{M}p_0 - 8\mathcal{M}p_1 + 3\mathcal{M}p_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 2x^2 + 9x + 1 & -6x^2 - 20x + 2 & x(7x+5) \\ x(7x+5) & -19x^2 - 6x + 1 & 15x^2 - 5x + 2 \\ 15x^2 - 5x + 2 & -38x^2 + 20x - 6 & 26x^2 - 21x + 7 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} f_{\mathcal{M}p_{2n+1}}(x) &= \frac{\mathcal{M}p_1 + (\mathcal{M}p_0 - 6\mathcal{M}p_1 + 3\mathcal{M}p_2)x + (3\mathcal{M}p_0 - 3\mathcal{M}p_1 + \mathcal{M}p_2)x^2}{1 - 3x + 3x^2 - x^3} \\ &= \frac{1}{1 - 3x + 3x^2 - x^3} \begin{pmatrix} 7x+5 & x^2 - 22x - 3 & 2x^2 + 9x + 1 \\ 2x^2 + 9x + 1 & -6x^2 - 20x + 2 & x(7x+5) \\ x(7x+5) & -19x^2 - 6x + 1 & 15x^2 - 5x + 2 \end{pmatrix}. \end{aligned}$$

3 Some Identities

In this section, we give some identities related to the generalized Guglielmo matrix sequences. Moreover, we deal with some special identities such as the Catalan's identity for the triangular, Lucas-triangular, oblong and pentagonal matrix sequences. First, we investigate relation, between $\mathcal{M}H_n$ and $\mathcal{M}W_n$ in the following theorem.

Theorem 3.1. *For any integer n, the following equalities are true:*

(a) $(W_0 - 2W_1 + W_2)\mathcal{M}H_n = 3\mathcal{M}W_{n+4} - 6\mathcal{M}W_{n+3} + 3\mathcal{M}W_{n+2}$.

(b) $(W_0 - 2W_1 + W_2)\mathcal{M}H_n = 3\mathcal{M}W_{n+3} - 6\mathcal{M}W_{n+2} + 3\mathcal{M}W_{n+1}$.

(c) $(W_0 - 2W_1 + W_2)\mathcal{M}H_n = 3\mathcal{M}W_{n+2} - 6\mathcal{M}W_{n+1} + 3\mathcal{M}W_n$.

(d) $(W_0 - 2W_1 + W_2)\mathcal{M}H_n = 3\mathcal{M}W_{n+1} - 6\mathcal{M}W_n + 3\mathcal{M}W_{n-1}$.

(e) $(W_0 - 2W_1 + W_2)\mathcal{M}H_n = 3\mathcal{M}W_n - 6\mathcal{M}W_{n-1} + 3\mathcal{M}W_{n-2}$.

Proof. For the prove identity (a), we use the mathematical induction on n. First we derive the proof for $n \geq 0$. If we take $n = 0$, the identity (a) is true. We assume that the identity (a) holds for $n \leq k$ and then ,for $n = k+1$ and using (2.2) and (1.6), we get

$$\begin{aligned} 3\mathcal{M}W_{(k+1)+4} - 6\mathcal{M}W_{(k+1)+3} + 3\mathcal{M}W_{(k+1)+2} &= \begin{pmatrix} 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 \\ 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 \\ 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 & 3W_0-6W_1+3W_2 \end{pmatrix} \\ &= (W_0 - 2W_1 + W_2)\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \\ &= (W_0 - 2W_1 + W_2)\mathcal{M}H_{k+1}. \end{aligned}$$

For the case $n < 0$, the proof can be done in the same way. Consequently, the other equalities can be proved similarly. Thus the proof completed. □

Next, the relations between generalized Guglielmo matrix sequences and oblong matrix sequences.

Theorem 3.2. *Let n be an arbitrary integer, then the following equalities are true:*

(a) $2\mathcal{M}W_n = (10W_0 - 15W_1 + 6W_2)\mathcal{M}O_{n+4} + (37W_1 - 24W_0 - 15W_2)\mathcal{M}O_{n+3} + (15W_0 - 24W_1 + 10W_2)\mathcal{M}O_{n+2}$.

(b) $2\mathcal{M}W_n = (6W_0 - 8W_1 + 3W_2)\mathcal{M}O_{n+3} + (21W_1 - 15W_0 - 8W_2)\mathcal{M}O_{n+2} + (10W_0 - 15W_1 + 6W_2)\mathcal{M}O_{n+1}$.

(c) $2\mathcal{M}W_n = (3W_0 - 3W_1 + W_2)\mathcal{M}O_{n+2} + (9W_1 - 8W_0 - 3W_2)\mathcal{M}O_{n+1} + (6W_0 - 8W_1 + 3W_2)\mathcal{M}O_n$.

(d) $2\mathcal{M}W_n = W_0O_{n+1} + (W_1 - 3W_0)\mathcal{M}O_n + (3W_0 - 3W_1 + W_2)\mathcal{M}O_{n-1}$.

(e) $2\mathcal{M}W_n = W_1O_n + (W_2 - 3W_1)\mathcal{M}O_{n-1} + W_0\mathcal{M}O_{n-2}$.

(f) $(W_0 - 2W_1 + W_2)^3\,\mathcal{M}O_n = -2(-3W_1^2 - W_2^2 + W_0W_1 + 3W_1W_2)\mathcal{M}W_{n+4} + 2(-8W_1^2 - 3W_2^2 + 3W_0W_1 - W_0W_2 + 9W_1W_2)\mathcal{M}W_{n+3} + 2(W_0^2 + 9W_1^2 + 3W_2^2 - 6W_0W_1 + 3W_0W_2 - 10W_1W_2)\mathcal{M}W_{n+2}$.

(g) $(W_0 - 2W_1 + W_2)^3\,\mathcal{M}O_n = -2(-W_1^2 + W_0W_2)\mathcal{M}W_{n+3} + 2(W_0^2 - 3W_0W_1 + 3W_0W_2 - W_1W_2)\mathcal{M}W_{n+2} - 2(-3W_1^2 - W_2^2 + W_0W_1 + 3W_1W_2)\mathcal{M}W_{n+1}$.

(h) $(W_0 - 2W_1 + W_2)^3\,\mathcal{M}O_n = 2(W_0^2 + 3W_1^2 - 3W_0W_1 - W_1W_2)\mathcal{M}W_{n+2} - 2(-W_2^2 + W_0W_1 - 3W_0W_2 + 3W_1W_2)\mathcal{M}W_{n+1} - 2(-W_1^2 + W_0W_2)\mathcal{M}W_n$.

(i) $(W_0 - 2W_1 + W_2)^3\,\mathcal{M}O_n = 2(3W_0^2 + 9W_1^2 + W_2^2 - 10W_0W_1 + 3W_0W_2 - 6W_1W_2)\mathcal{M}W_{n+1} - 2(3W_0^2 + 8W_1^2 - 9W_0W_1 + W_0W_2 - 3W_1W_2)\mathcal{M}W_n + 2(W_0^2 + 3W_1^2 - 3W_0W_1 - W_1W_2)\mathcal{M}W_{n-1}$.

(j) $(W_0 - 2W_1 + W_2)^3\,\mathcal{M}O_n = 2(6W_0^2 + 19W_1^2 + 3W_2^2 - 21W_0W_1 + 8W_0W_2 - 15W_1W_2)\mathcal{M}W_n - 2(8W_0^2 + 24W_1^2 + 3W_2^2 - 27W_0W_1 + 9W_0W_2 - 17W_1W_2)\mathcal{M}W_{n-1} + 2(3W_0^2 + 9W_1^2 + W_2^2 - 10W_0W_1 + 3W_0W_2 - 6W_1W_2)\mathcal{M}W_{n-2}$.

For the prove identity (a), we use the mathematical induction on n. First we derive the proof for $n \geq 0$. If we take $n = 0$ the identity (a) is true. We assume that the identity (a) holds for $n \leq k$ and then ,for $n = k+1$ and using (2.2) and (1.6), we get

$$(10W_0-15W_1+6W_2)\mathcal{M}O_{(k+1)+4}+(37W_1-24W_0-15W_2)\mathcal{M}O_{(k+1)+3}+(15W_0-24W_1+10W_2)\mathcal{M}O_{(k+1)+2}=\begin{pmatrix} 2W_{n+1} & -6W_n+2W_{n-1} & 2W_n \\ 2W_n & -6W_{n-1}+2W_{n-2} & 2W_{n-1} \\ 2W_{n-1} & -6W_{n-2}+2W_{n-3} & 2W_{n-2} \end{pmatrix}=2\mathcal{M}W_{k+1}.$$

For the case $n<0$, the proof can be done in the same way. Consequently the other equalities can be proved similarly. Thus the proof completed. □

Next, the relations between generalized Guglielmo Matrix Sequences and pentagonal Matrix Sequences have been given.

Theorem 3.3. *For any integer n, the following equalities are true:*

(a) $27\mathcal{M}W_n=(64W_0-89W_1+34W_2)\mathcal{M}p_{n+4}+(229W_1-158W_0-89W_2)\mathcal{M}p_{n+3}+(103W_0-158W_1+64W_2)\mathcal{M}p_{n+2}$.

(b) $27\mathcal{M}W_n=(34W_0-38W_1+13W_2)\mathcal{M}p_{n+3}+(109W_1-89W_0-38W_2)\mathcal{M}p_{n+2}+(64W_0-89W_1+34W_2)\mathcal{M}p_{n+1}$.

(c) $27\mathcal{M}W_n=(13W_0-5W_1+W_2)\mathcal{M}p_{n+2}+(25W_1-38W_0-5W_2)\mathcal{M}p_{n+1}+(34W_0-38W_1+13W_2)\mathcal{M}p_n$.

(d) $27\mathcal{M}W_n=(W_0+10W_1-2W_2)\mathcal{M}p_{n+1}+(10W_2-23W_1-5W_0)\mathcal{M}p_n+(13W_0-5W_1+W_2)\mathcal{M}p_{n-1}$.

(e) $27\mathcal{M}W_n=(7W_1-2W_0+4W_2)\mathcal{M}p_n+(10W_0-35W_1+7W_2)\mathcal{M}p_{n-1}+(W_0+10W_1-2W_2)\mathcal{M}p_{n-2}$.

(f) $(W_0-2W_1+W_2)^3\mathcal{M}p_n=(2W_0^2+21W_1^2+7W_2^2-13W_0W_1+6W_0W_2-23W_1W_2)\mathcal{M}W_{n+4}+(-6W_0^2+37W_0W_1-19W_0W_2-56W_1^2+63W_1W_2-19W_2^2)\mathcal{M}W_{n+3}+(7W_0^2+47W_1^2+15W_2^2-36W_0W_1+19W_0W_2-52W_1W_2)\mathcal{M}W_{n+2}$.

(g) $(W_0-2W_1+W_2)^3\mathcal{M}p_n=(7W_1^2+2W_2^2-2W_0W_1-W_0W_2-6W_1W_2)\mathcal{M}W_{n+3}+(W_0^2-16W_1^2-6W_2^2+3W_0W_1+W_0W_2+17W_1W_2)\mathcal{M}W_{n+2}+(2W_0^2+21W_1^2+7W_2^2-13W_0W_1+6W_0W_2-23W_1W_2)\mathcal{M}W_{n+1}$.

(h) $(W_0-2W_1+W_2)^3\mathcal{M}p_n=(W_0^2+5W_1^2-3W_0W_1-2W_0W_2-W_1W_2)\mathcal{M}W_{n+2}+(2W_0^2+W_2^2-7W_0W_1+9W_0W_2-5W_1W_2)\mathcal{M}W_{n+1}+(7W_1^2+2W_2^2-2W_0W_1-W_0W_2-6W_1W_2)\mathcal{M}W_n$.

(i) $(W_0-2W_1+W_2)^3\mathcal{M}p_n=(5W_0^2+15W_1^2+W_2^2-16W_0W_1+3W_0W_2-8W_1W_2)W_{n+1}+(-3W_0^2+7W_0W_1+5W_0W_2-8W_1^2-3W_1W_2+2W_2^2)\mathcal{M}W_n+(W_0^2+5W_1^2-3W_0W_1-2W_0W_2-W_1W_2)\mathcal{M}W_{n-1}$.

(j) $(W_0-2W_1+W_2)^3\mathcal{M}p_n=(12W_0^2+37W_1^2+5W_2^2-41W_0W_1+14W_0W_2-27W_1W_2)\mathcal{M}W_n+(-14W_0^2+45W_0W_1-11W_0W_2-40W_1^2+23W_1W_2-3W_2^2)\mathcal{M}W_{n-1}+(5W_0^2+15W_1^2+W_2^2-16W_0W_1+3W_0W_2-8W_1W_2)\mathcal{M}W_{n-2}$.

For the prove identity (a), we use the mathematical induction on n. First, we derive the proof for $n\geq 0$. If we take $n=0$ the identity (a) is true. We assume that the identity (a) holds for $n\leq k$ and then ,for $n=k+1$ and using (2.2) and (1.6), we get,

$$(64W_0-89W_1+34W_2)\mathcal{M}p_{k+5}+(229W_1-158W_0-89W_2)\mathcal{M}p_{k+4}+(103W_0-158W_1+64W_2)\mathcal{M}p_{k+3}=\begin{pmatrix} 27W_{k+2} & 27W_k & 27W_{k+1} \\ 27W_{k+1} & 27W_{k-1} & 27W_k \\ 27W_k & 27W_{k-2} & 27W_{k-1} \end{pmatrix}=27\mathcal{M}W_{k+1}.$$

For the case $n<0$, the proof can be done in the same way. Consequently, the other equalities can be prove similarly. Thus the proof completed. □

Theorem 3.4. *For all integers m,n the following identity holds:*

$$\mathcal{M}W_{m+n}=T_{m-1}\mathcal{M}W_{n+2}+(T_{m-3}-3T_{m-2})\mathcal{M}W_{n+1}+T_{m-2}\mathcal{M}W_n.$$

Proof. First, we assume that $m, n \geqslant 0$. the theorem (3.4) can be proved by mathematical induction on m. If $m = 0$ we get

$$\mathcal{M}W_n = T_{-1}\mathcal{M}W_{n+2} + (T_{-3} - 3T_{-2})\mathcal{M}W_{n+1} + T_{-2}\mathcal{M}W_n$$

which is true since $T_{-1} = 0, T_{-2} = 1, T_{-3} = 3$. We assume that the identity given holds for $m \leq k$. For $m = k+1$, we get

$$\begin{aligned}
\mathcal{M}W_{(k+1)+n} &= 3\mathcal{M}W_{n+k} - 3\mathcal{M}W_{n+k-1} + \mathcal{M}W_{n+k-2} \\
&= 3(T_{k-1}\mathcal{M}W_{n+2} + (T_{k-3} - 3T_{k-2})\mathcal{M}W_{n+1} + T_{k-2}\mathcal{M}W_n) \\
&\quad -3(T_{k-2}\mathcal{M}W_{n+2} + (T_{k-4} - 3T_{k-3})\mathcal{M}W_{n+1} + T_{k-3}\mathcal{M}W_n) \\
&\quad +(T_{k-3}\mathcal{M}W_{n+2} + (T_{k-5} - 3T_{k-4})\mathcal{M}W_{n+1} + T_{k-4}\mathcal{M}W_n) \\
&= (3T_{k-1} - 3T_{k-2} + T_{k-3})\mathcal{M}W_{n+2} + ((3T_{k-3} - 3T_{k-4} + T_{k-5}) \\
&\quad -3(3T_{k-2} - 3T_{k-3} + T_{k-4}))\mathcal{M}W_{n+1} + (3T_{k-2} - 3T_{k-3} + T_{k-4})\mathcal{M}W_n \\
&= T_k\mathcal{M}W_{n+2} + (T_{k-2} - 3T_{k-1})\mathcal{M}W_{n+1} + T_{k-1}\mathcal{M}W_n \\
&= T_{(k+1)-1}\mathcal{M}W_{n+2} + (T_{(k+1)-3} - 3T_{(k+1)-2})\mathcal{M}W_{n+1} + T_{(k+1)-2}\mathcal{M}W_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 3.4. The case $m, n < 0$ can be proved similarly. □

For $n \geq 0$, $m \geq 0$ and taking $\mathcal{M}W_n = \mathcal{M}T_n$ or $\mathcal{M}W_n = \mathcal{M}H_n$ or $\mathcal{M}W_n = \mathcal{M}O_n$ or $\mathcal{M}W_n = \mathcal{M}p_n$, respectively, we get,

$$\begin{aligned}
\mathcal{M}T_{m+n} &= T_{m-1}\mathcal{M}T_{n+2} + (T_{m-3} - 3T_{m-2})\mathcal{M}T_{n+1} + T_{m-2}\mathcal{M}T_n, \\
\mathcal{M}H_{m+n} &= T_{m-1}\mathcal{M}H_{n+2} + (T_{m-3} - 3T_{m-2})\mathcal{M}H_{n+1} + T_{m-2}\mathcal{M}H_n, \\
\mathcal{M}O_{m+n} &= T_{m-1}\mathcal{M}O_{n+2} + (T_{m-3} - 3T_{m-2})\mathcal{M}O_{n+1} + T_{m-2}\mathcal{M}O_n, \\
\mathcal{M}p_{m+n} &= T_{m-1}\mathcal{M}p_{n+2} + (T_{m-3} - 3T_{m-2})\mathcal{M}p_{n+1} + T_{m-2}\mathcal{M}p_n.
\end{aligned}$$

Next Lemma, we give some identities raleted to $\mathcal{M}A_1$, $\mathcal{M}A_2$, $\mathcal{M}A_3$, $\mathcal{M}B_1$, $\mathcal{M}B_2$, $\mathcal{M}B_3$, $\mathcal{M}C_1$, $\mathcal{M}C_2$, $\mathcal{M}C_3$, $\mathcal{M}D_1$, $\mathcal{M}D_2$, $\mathcal{M}D_3$, $\mathcal{M}E_1$, $\mathcal{M}E_2$ and $\mathcal{M}E_3$.

Lemma 3.5. *The following identities holds:*

$$\begin{aligned}
\mathcal{M}B_1\mathcal{M}A_1 &= \mathcal{M}B_1\mathcal{M}C_1 = \mathcal{M}B_1\mathcal{M}D_1 = \mathcal{M}B_1\mathcal{M}E_1 \\
\mathcal{M}B_3^2 &= \mathcal{M}B_2\mathcal{M}B_3 = \mathcal{M}B_3\mathcal{M}B_2 = 0 \\
\mathcal{M}B_2\mathcal{M}C_2 &= \mathcal{M}C_2\mathcal{M}B_2 = \mathcal{M}E_2\mathcal{M}C_2 = \mathcal{M}C_2\mathcal{M}E_2 = 0 \\
\mathcal{M}B_3\mathcal{M}E_3 &= \mathcal{M}E_3\mathcal{M}B_3 = \mathcal{M}B_3\mathcal{M}D_3 = \mathcal{M}D_3\mathcal{M}B_3 = 0
\end{aligned}$$

Proof. Using matrix multiplication, the proof is done. □

Note that $\mathcal{M}B_1 = \mathcal{M}T_0$ is identity matrix.

Next Lemma, we give some relation linked to Guglielmo matrix sequences.

Lemma 3.6. *For all integers m and n, we have the following identities.*

(a) $\mathcal{M}T_0\mathcal{M}W_n = \mathcal{M}W_n\mathcal{M}T_0 = \mathcal{M}W_n$.

(b) $\mathcal{M}W_0\mathcal{M}T_n = \mathcal{M}T_n\mathcal{M}W_0 = \mathcal{M}W_n$.

(c) $\mathcal{M}T_n\mathcal{M}T_m = \mathcal{M}T_m\mathcal{M}T_n = \mathcal{M}T_{n+m}$.

(d) $\mathcal{M}T_m\mathcal{M}W_n = \mathcal{M}W_n\mathcal{M}T_m = \mathcal{M}W_{m+n}$.

(e) $\mathcal{M}T_m\mathcal{M}H_n = \mathcal{M}H_n\mathcal{M}T_m = \mathcal{M}H_{m+n}$.

(f) $\mathcal{M}T_m\mathcal{M}O_n = \mathcal{M}O_n\mathcal{M}T_m = \mathcal{M}O_{m+n}$.

(g) $\mathcal{M}T_m\mathcal{M}p_n = \mathcal{M}p_n\mathcal{M}T_m = \mathcal{M}p_{m+n}$.

(h) $\mathcal{M}W_0\mathcal{M}W_n = \mathcal{M}W_n\mathcal{M}W_0$.

(i) $\mathcal{M}W_n\mathcal{M}W_m = \mathcal{M}W_m\mathcal{M}W_n = \mathcal{M}W_0\mathcal{M}W_{n+m}$.

(j) $\mathcal{M}T_{-n} = (\mathcal{M}T_n)^{-1}$.

(k) $\mathcal{M}W_{-n} = (\mathcal{M}W_0)^{1-n}(\mathcal{M}W_{-1})^n$.

Proof.

(a) The proof can be easily seen since $\mathcal{M}T_0$ is the identity matrix.

(b) The proof can be seen by using Theorem 2.1 and Corollary 2.2.

(c) Using Lemma 3.5 we have the following equalities,

$$\begin{aligned}
\mathcal{M}T_n\mathcal{M}T_m &= (\mathcal{M}B_1 + \mathcal{M}B_2n + \mathcal{M}B_3n^2)(\mathcal{M}B_1 + \mathcal{M}B_2m + \mathcal{M}B_3m^2) \\
&= \mathcal{M}B_1^2 + \mathcal{M}B_1\mathcal{M}B_2m + \mathcal{M}B_1\mathcal{M}B_3m^2 + \mathcal{M}B_1\mathcal{M}B_2n + \mathcal{M}B_2\mathcal{M}B_2mn + \\
&= \mathcal{M}B_2\mathcal{M}B_3nm^2 + \mathcal{M}B_1\mathcal{M}B_3n^2 + \mathcal{M}B_3\mathcal{M}B_2mn^2 + \mathcal{M}B_3\mathcal{M}B_3m^2n^2 \\
&= (\mathcal{M}B_1 + \mathcal{M}B_2(n+m) + \mathcal{M}B_3(n+m)^2) \\
&= \mathcal{M}T_{n+m}.
\end{aligned}$$

The identity $\mathcal{M}T_m\mathcal{M}T_n = \mathcal{M}T_{n+m}$ can be seen similarly. This completes the proof that we need.

(d) From (b) and (c), we have

$$\begin{aligned}
\mathcal{M}T_m\mathcal{M}W_n &= \mathcal{M}T_m\mathcal{M}T_n\mathcal{M}W_0 \\
&= \mathcal{M}T_{n+m}\mathcal{M}W_0 \\
&= \mathcal{M}W_{m+n}.
\end{aligned}$$

(e) Take $\mathcal{M}W_n = \mathcal{M}H_n$ in (d).

(f) Take $\mathcal{M}W_n = \mathcal{M}O_n$ in (d).

(g) Take $\mathcal{M}W_n = \mathcal{M}p_n$ in (d).

(h) If we perform matrix multiplication and compare the entries of the resulting rows and columns, the required proof is complete.

(i) If we use (d) and (h) and (b), we have

$$\begin{aligned}
\mathcal{M}W_0\mathcal{M}W_{n+m} &= \mathcal{M}W_0\mathcal{M}W_n\mathcal{M}T_m \\
&= \mathcal{M}W_n\mathcal{M}W_0\mathcal{M}T_m \\
&= \mathcal{M}W_n\mathcal{M}W_m
\end{aligned}$$

The identity $\mathcal{M}W_0\mathcal{M}W_{n+m} = \mathcal{M}W_m\mathcal{M}W_n$ can be seen similarly. This completes the proof of (h).

(j) First, we establish the proof the case $n \geq 0$ by using mathematical induction on n. If $n = 0$ then we have

$$\mathcal{M}T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = (\mathcal{M}T_0)^{-1}$$

and

$$\mathcal{M}T_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}^{-1} = (\mathcal{M}T_1)^{-1}.$$

Let the equality given in the (j) holds for $n \leq k$. For $n = k+1$, by using (c), we obtain

$$\begin{aligned}
(\mathcal{M}T_{k+1})^{-1} &= (\mathcal{M}T_k\mathcal{M}T_1)^{-1} = (\mathcal{M}T_1)^{-1}(\mathcal{M}T_k)^{-1} = \mathcal{M}T_{-1}\mathcal{M}T_{-k} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} T_{-n+1} & -3T_{-n} + T_{-n-1} & T_{-n} \\ T_{-n} & -3T_{-n-1} + T_{-n-2} & T_{-n-1} \\ T_{-n-1} & -3T_{-n-2} + T_{-n-3} & T_{-n-2} \end{pmatrix} \\
&= \begin{pmatrix} T_{-k} & -3T_{-k-1} + T_{-k-2} & T_{-k-1} \\ T_{-k-1} & -3T_{-k-2} + T_{-k-3} & T_{-k-2} \\ 3T_{-k-1} - 3T_{-k} + T_{-k+1} & 10T_{-k-1} - 12T_{-k-2} + 3T_{-k-3} - 3T_{-k} & 3T_{-k-2} - 3T_{-k-1} + T_{-k} \end{pmatrix} \\
&= \begin{pmatrix} T_{-(k+1)+1} & -3T_{-(k+1)} + T_{-(k+1)-1} & T_{-(k+1)} \\ T_{-(k+1)} & -3T_{-(k+1)-1} + T_{-(k+1)-2} & T_{-(k+1)-1} \\ T_{-(k+1)-1} & -3T_{-(k+1)-2} + T_{-(k+1)-3} & T_{-(k+1)-2} \end{pmatrix} \\
&= \mathcal{M}T_{-(k+1)}.
\end{aligned}$$

So we get the result that we need. For the other case $n < 0$, the proof can be done similarly. This complates proof.

(k) If we take $m = -n+1$ and $n = -1$ in the identity (i), we obtain that

$$\mathcal{M}W_0\mathcal{M}W_{-n} = \mathcal{M}W_{-n+1}\mathcal{M}W_{-1}. \tag{3.1}$$

If we multiply both side of the equation that we obtain above with $\mathcal{M}W_0$ and using (i) we have the relation

$$\begin{aligned}
\mathcal{M}W_0\mathcal{M}W_0\mathcal{M}W_{-n} &= \mathcal{M}W_0\mathcal{M}W_{-n+1}\mathcal{M}W_{-1} \\
&= \mathcal{M}W_{-n+1}\mathcal{M}W_{-1}\mathcal{M}W_{-1}.
\end{aligned}$$

Repeating this process, we obtain

$$(\mathcal{M}W_0)^{n-1}\mathcal{M}W_{-n} = (\mathcal{M}W_{-1})^n.$$

Thus, it follows that

$$\mathcal{M}W_{-n} = (\mathcal{M}W_0)^{1-n}(\mathcal{M}W_{-1})^n.$$

This completes the proof. □

Note that using Lemma 3.6 (k) and (d), we obtain

$$\begin{aligned}
\mathcal{M}W_{-n} &= (\mathcal{M}W_0)^{1-n}(\mathcal{M}W_{-1})^n \\
&= (\mathcal{M}W_n\mathcal{M}W_{-n})^{1-n}(\mathcal{M}W_{-1})^n \\
&= (\mathcal{M}W_{-n})^{1-n}(\mathcal{M}W_n)^{1-n}(\mathcal{M}W_{-1})^n
\end{aligned}$$

and then by Lemma (j), we get

$$\mathcal{M}W_{-n} = (\mathcal{M}T_n)^{1-n}(\mathcal{M}W_n)^{1-n}(\mathcal{M}W_{-1})^n.$$

Using Lemma 3.6 and comparing matrix entries, we obtain the next result.

Theorem 3.7. *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{M}T_n^m = \mathcal{M}T_{mn}$,

(b) $\mathcal{M}T_{n+1}^m = \mathcal{M}T_1^m \mathcal{M}T_{mn}$,

(c) $\mathcal{M}T_{n-r}\mathcal{M}T_{n+r} = \mathcal{M}T_n^2 = \mathcal{M}T_2^n$.

Proof. We prove for $m, n, r \geq 0$. The other cases can be proved similarly.

(a) We can write $\mathcal{N}_n^m$ as

$$\mathcal{M}T_n^m = \mathcal{M}T_n \mathcal{M}T_n ... \mathcal{M}T_n \ (m \text{ times}).$$

Using Lemma 3.6 (c) iteratively, we obtain the required result:

$$\begin{aligned} \mathcal{M}T_n^m &= \underbrace{\mathcal{M}T_n \mathcal{M}T_n ... \mathcal{M}T_n}_{m \text{ times}} \\ &= \mathcal{M}T_{2n} \underbrace{\mathcal{M}T_n \mathcal{M}T_n ... \mathcal{M}T_n}_{m-1 \text{ times}} \\ &= \mathcal{M}T_{3n} \underbrace{\mathcal{M}T_n \mathcal{M}T_n ... \mathcal{M}T_n}_{m-2 \text{ times}} \\ &\vdots \\ &= \mathcal{M}T_{(m-1)n} \mathcal{M}T_n \\ &= \mathcal{M}T_{mn}. \end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{M}T_{n+1}^m = \mathcal{M}T_{n+1}.\mathcal{M}T_{n+1} ... \mathcal{M}T_{n+1} = \mathcal{M}T_{m(n+1)} = \mathcal{M}T_m \mathcal{M}T_{mn} = \mathcal{M}T_1 \mathcal{M}T_{m-1} \mathcal{M}T_{mn}.$$

Using Lemma 3.6 (c), we can write iteratively,

$$\begin{aligned} \mathcal{M}T_m &= \mathcal{M}T_1 \mathcal{M}T_{m-1}, \\ \mathcal{M}T_{m-1} &= \mathcal{M}T_1 \mathcal{M}T_{m-2}, \\ &\vdots \\ \mathcal{M}T_2 &= \mathcal{M}T_1 \mathcal{M}T_1. \end{aligned}$$

Now it follows that

$$\begin{aligned} \mathcal{M}T_{n+1}^m &= \mathcal{M}T_1 \mathcal{M}T_{m-1} \mathcal{M}T_{mn} \\ &= \underbrace{\mathcal{M}T_1 \mathcal{M}T_1 ... \mathcal{M}T_1}_{m \text{ times}} \mathcal{M}T_{mn} = \mathcal{M}T_1^m \mathcal{M}T_{mn}. \end{aligned}$$

(c) Lemma 3.6 (c) gives

$$\mathcal{M}T_{n-r}\mathcal{M}T_{n+r} = \mathcal{M}T_{2n} = \mathcal{M}T_n \mathcal{M}T_n = \mathcal{M}T_n^2$$

and also

$$\mathcal{M}T_{n-r}\mathcal{M}T_{n+r} = \mathcal{M}T_{2n} = \underbrace{\mathcal{M}T_2 \mathcal{M}T_2 ... \mathcal{M}T_2}_{n \text{ times}} = \mathcal{M}T_2^n.$$

Next Theorem, we encounter similar outcomes for the matrix sequence $\mathcal{M}W_n$.

Theorem 3.8. *For all integers m, n and r, the following identities hold:*

(a) $\mathcal{M}W_{n-r}\mathcal{M}W_{n+r} = \mathcal{M}W_n^2$,

(b) $\mathcal{M}W_n^m = \mathcal{M}W_0^m \mathcal{M}T_{mn}$.

Proof.

(a) Using Binet's formula of generalized Guglielmo sequence given in Theorem 2.4 and matrix multiplication, we obtain

$$\begin{aligned}\mathcal{M}W_{n-r}\mathcal{M}W_{n+r} - \mathcal{M}W_n^2 &= (\mathcal{M}A_1 + \mathcal{M}A_2(n+r) + \mathcal{M}A_3(n+r)^2)(\mathcal{M}A_1 + \mathcal{M}A_2(n-r) \\ &\quad + \mathcal{M}A_3(n-r)^2) - (\mathcal{M}A_1 + \mathcal{M}A_2 n + \mathcal{M}A_3 n^2)^2 \\ &= -2n^2r^2\mathcal{M}A_3^2 - 2nr^2\mathcal{M}A_2\mathcal{M}A_3 + r^4\mathcal{M}A_3^2 - r^2\mathcal{M}A_2^2 + 2\mathcal{M}A_1 r^2\mathcal{M}A_3 \\ &= 0.\end{aligned}$$

At least, we get the result as required.

(b) By Theorem 3.7, we have

$$\mathcal{M}W_0^m\mathcal{M}T_{mn} = \underbrace{\mathcal{M}W_0\mathcal{M}W_0...\mathcal{M}W_0}_{m \text{ times}}\underbrace{\mathcal{M}T_n\mathcal{M}T_n...\mathcal{M}T_n}_{m \text{ times}}.$$

By applying Lemma 3.6 (b) iteratively, it follows that

$$\begin{aligned}\mathcal{M}W_0^m\mathcal{M}T_{mn} &= (\mathcal{M}W_0\mathcal{M}T_n)(\mathcal{M}W_0\mathcal{M}T_n)...(\mathcal{M}W_0\mathcal{M}T_n) \\ &= \mathcal{M}W_n\mathcal{M}W_n...\mathcal{M}W_n = \mathcal{M}W_n^m.\end{aligned}$$

This completes the proof. □

4 Conclusion

In thıs study we define generalized Guglielmo matrix sequence and give four special cases that we named triangular matrix sequence, Lucas-triangular matrix sequence, oblong matrix sequence and pentagonals matrix sequence. We also define the Binet's formula of the generalized Guglielmo matrix sequence and its special cases. We give generating function and sum formulas related to generalized Guglielmo matrix sequence. Furthermore, we give some lemma and theorem using matrix properties and some formulas. We know that the matrix sequences are fundamental issue in linear algebra and have applications in various fields including numerical analysis, differential equations, control theory, and computer graphics. Thus, we believe that our study will contribute to ongoing research in this field.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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