Nonlinear Saturation Controllers for Suppression the Vibration of Nonlinear Spring Pendulum

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this work, we apply the Averaging Method to obtain the theoretical results. The Nonlinear Saturation Controller (NSC) is proposed to decrease the vacillations of the spring pendulum. We investigate the stability of the system nigh the resonance condition by applying the frequency response equations. Numerically, the effects of diversified controller's parameters on the basic system behaviour are studied. The emulation results are attained by utilising Matlab and Maple programs.

1 Introduction

One of the most important engineering problems is vibration control. Many methods have been developed for suppression the vibration using active or passive controllers. It is worth to be mentioned that several control techniques are used to regulate the reaction of a system such like: passive control, semi active

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control, active control, and hybrid control. In the work [1], Kamel et al presented the vacillation and stability of the nonlinear spring pendulum which was depicting the roll motion of a ship. They studied the influences of the linear controller on the fundamental system subject to multi parametric stimulation. Zhou and Chen [2] used two procedures of ship example under sinusoidal harmonic stimulation to investigate the response and constancy of the system. They got the equation of bifurcation response close to the collection resonance case in the existence of internal resonance of the studied system.

Lee et al. [3-5] studied the demeanor of the Spring Pendulum system subordinated to single-harmonic-excitation force. The concluded results elucidated that the system had an extremely complicated attitude which include Hopf bifurcations and jump phenomena. Not that only, they also discovered that the approximation with the 2nd - order gave more perfect harmonisation with the fundamental system than the 1st – order did. Song et al. [6] scrutinised the oscillation reaction of the spring mass damper system with an agitated parametrically pendulum suspended to the mass by using the harmonic balance method. Furthermore, they illustrated the unstable motion zone of the system which gotten from the 3rd - order approximation to become completely coordinated with which acquired from numeral computation. Eissa et al [7–10] and Sayed [11] applied diversified controllers on the simple and the spring pendulums which showing the sway motion of the ship then; they studied the effects of them. They subjected the two procedures under single-harmonic- stimulation force then they studied the action of the transversal and linear toned absorbed controllers on the existing vibrations. Structural damping treatment is one of the exemplary passive vibrations control near that had been utilised in the practical structural engineering, but active damping had also enticed the interesting of a considerable number of investigators [12–14].

The researchers in the work [15], EL-Sayed and Bauomy subjected a nonlinear dynamic system to multi-parametric-stimulation forces then they investigated the effectiveness of time- delay controller on the existent vibrations. And they applied the averaging method to drive the system's frequency response equations. Hamed and Amer [16] introduced a research for (NSC) controller which had been utilised to repress the vibrated amplitude of a structural dynamic model assuming non- linear composite beam. They had a great result by utilising the saturation for frustrating the vacillation of their non-linear system. Warminski J. et al. [17] introduced experimental and numerical surveys of four kinds of controllers utilised to non- linear beam models. Validation of distinct control designs is estimated utilising numeral simulations in Matlab-Simulink program software. The mathematical results for non- linear dynamic system with NSC are acquired utilising multiple scale method. The numerical and analytical interpretation for NSC submits the effect of most substantial parameters on the efficiency of the control of a non- linear established paradigm for an extensive domain of frequency of stimulation and peak scale of amplitude. The consequences for a solo beam system display which positive position feedback (PPF) and (NSC) controllers are the most influential for supposed provisions of the established paradigm.

Saeed et al. [18] used saturation-based controller with time delay in active extinction of nonlinear beam vibrations. They found that the time delays append to the response neoteric control keys through contraction the bandwidth of the controller's frequency. They mathematical solutions were in a great agreement with the numerical ones. Saeed and El-Gohary [19] modified the model of saturation controllers which added to the system through quadratic nonlinearity coupling. They investigated the leverage of time delays on the system's stability and controller's behavior. Ashour and Nayfeh [20] had used a non- linear controller as a vacillation suppressor established on saturation incident influential for depression of torsional and flexural oscillations of the plate. The assessment of hesitancy of stimulation had been appended to the system to magnify the leverage of the saturation control.

Our main aim of the present paper is to apply control to diminish the vacillation of the system. This controller is NSC control. Averaging technique is used to get the mathematical analysis. Some testaments concerning the diversified parameters of the spring pendulum are notified and the influences of the NSC controllers on the system's demeanor will be studied numerically.
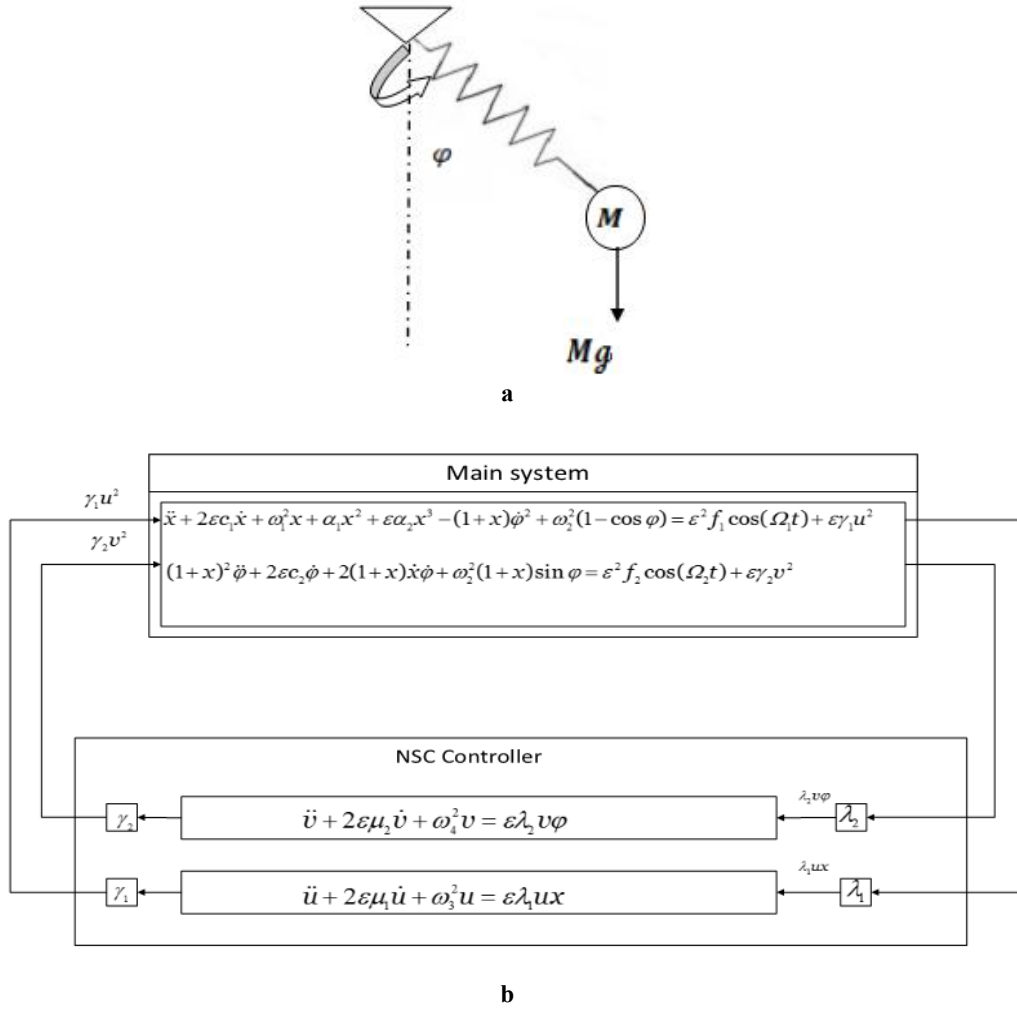
a

b

Fig. 1. (a) Model of a spring pendulum   (b) Block diagram of the system

2 Mathematical Model

The differential equations of motion characterising the oscillations of the non- linear spring pendulum with NSC controllers could be expressed as the next form:

$$\ddot{x} + 2\varepsilon c_1\dot{x} + \omega_1^2 x + \alpha_1 x^2 + \varepsilon\alpha_2 x^3 - (1+x)\dot{\varphi}^2 + \omega_2^2(1-\cos\varphi) = \varepsilon^2 f_1\cos(\Omega_1 t) + \varepsilon\gamma_1 u^2 \quad (1)$$

$$(1+x)^2\ddot{\varphi} + 2\varepsilon c_2\dot{\varphi} + 2(1+x)\dot{x}\dot{\varphi} + \omega_2^2(1+x)\sin\varphi = \varepsilon^2 f_2\cos(\Omega_2 t) + \varepsilon\gamma_2 v^2 \quad (2)$$

$$\ddot{u} + 2\varepsilon\mu_1\dot{u} + \omega_3^2 u = \varepsilon\lambda_1 ux \quad (3)$$

$$\ddot{v} + 2\varepsilon\mu_2\dot{v} + \omega_4^2 v = \varepsilon\lambda_2 v\varphi \quad (4)$$

Where x and φ are the longitudinal and angular response of the spring pendulum respectively. u and v known as the response of the absorbers. c_j, μ_j ($j=1,2$) are the spring pendulum modes' damping coefficients and the NSC controllers, respectively. $\omega_1, \omega_2, \omega_3$ and ω_4 are the spring pendulum modes' natural frequencies of and NSC controllers, respectively. α_1 and α_2 are the non-linear parameters f_j is the external forcing amplitudes of the fundamental system Ω_j is the excitation frequencies of the fundamental system γ_1, γ_2 are the control signal gains and λ_1, λ_2 are the feedback signal gains.

3 Mathematical Analysis (Averaging Method)

Applying the averaging method, the solutions for equations (1-4) from the first-order approximate are considering as:

$$x = a_1 \cos\left(\omega_1 t + \psi_1\right) \tag{5a}$$

$$\varphi = a_2 \cos\left(\omega_2 t + \psi_2\right) \tag{5b}$$

$$u = a_3 \cos\left(\omega_3 t + \psi_3\right) \tag{5c}$$

$$v = a_4 \cos\left(\omega_4 t + \psi_4\right) \tag{5d}$$

Where a_n and ψ_n, $(n=1,2,3,4)$ exist as constants. Differentiate equations (5a)- (5d) we get:

$$\dot{x} = -a_1\omega_1 \sin\left(\omega_1 t + \psi_1\right) \tag{6a}$$

$$\dot{\varphi} = -a_2\omega_2 \sin\left(\omega_2 t + \psi_2\right) \tag{6b}$$

$$\dot{u} = -a_3\omega_3 \sin\left(\omega_3 t + \psi_3\right) \tag{6c}$$

$$\dot{v} = -a_4\omega_4 \sin\left(\omega_4 t + \psi_4\right) \tag{6d}$$

For $\varepsilon \neq 0$, relatively small, claim a_n and ψ_n, $(n=1,2,3,4)$ are dependent variables on time t in equations (1) - (4). Differentiating equations (5a) - (5d) once with respect to time t submits

$$\dot{x} = \dot{a}_1 \cos\left(\omega_1 t + \psi_1\right) - a_1\omega_1 \sin\left(\omega_1 t + \psi_1\right) - a_1\dot{\psi}_1 \sin\left(\omega_1 t + \psi_1\right) \tag{7a}$$

$$\dot{\varphi} = \dot{a}_2 \cos\left(\omega_2 t + \psi_2\right) - a_2\omega_2 \sin\left(\omega_2 t + \psi_2\right) - a_2\dot{\psi}_2 \sin\left(\omega_2 t + \psi_2\right) \tag{7b}$$

$$\dot{u} = \dot{a}_3 \cos\left(\omega_3 t + \psi_3\right) - a_3\omega_3 \sin\left(\omega_3 t + \psi_3\right) - a_3\dot{\psi}_3 \sin\left(\omega_3 t + \psi_3\right) \tag{7c}$$

$$\dot{v} = \dot{a}_4 \cos\left(\omega_4 t + \psi_4\right) - a_4\omega_4 \sin\left(\omega_4 t + \psi_4\right) - a_4\dot{\psi}_4 \sin\left(\omega_4 t + \psi_4\right) \tag{7d}$$

Comparing equations (6a) - (6d) and equations (7a) - (7d) we realise that:

$$\dot{a}_1 \cos(\omega_1 t + \psi_1) - a_1 \dot{\psi}_1 \sin(\omega_1 t + \psi_1) = 0$$

$$\dot{a}_2 \cos(\omega_2 t + \psi_2) - a_2 \dot{\psi}_2 \sin(\omega_2 t + \psi_2) = 0$$

$$\dot{a}_3 \cos(\omega_3 t + \psi_3) - a_3 \dot{\psi}_3 \sin(\omega_3 t + \psi_3) = 0$$

$$\dot{a}_4 \cos(\omega_4 t + \psi_4) - a_4 \dot{\psi}_4 \sin(\omega_4 t + \psi_4) = 0$$

Differentiating equations (5a) - (5d) once respect to time t we find:

$$\ddot{x} = -\dot{a}_1 \omega_1 \sin(\omega_1 t + \psi_1) - a_1 \omega_1^2 \cos(\omega_1 t + \psi_1) - a_1 \omega_1 \dot{\psi}_1 \cos(\omega_1 t + \psi_1) \tag{8a}$$

$$\ddot{\varphi} = -\dot{a}_2 \omega_2 \sin(\omega_2 t + \psi_2) - a_2 \omega_2^2 \cos(\omega_2 t + \psi_2) - a_2 \omega_2 \dot{\psi}_2 \cos(\omega_2 t + \psi_2) \tag{8b}$$

$$\ddot{u} = -\dot{a}_3 \omega_3 \sin(\omega_3 t + \psi_3) - a_3 \omega_3^2 \cos(\omega_3 t + \psi_3) - a_3 \omega_3 \dot{\psi}_3 \cos(\omega_3 t + \psi_3) \tag{8c}$$

$$\ddot{v} = -\dot{a}_4 \omega_4 \sin(\omega_4 t + \psi_4) - a_4 \omega_4^2 \cos(\omega_4 t + \psi_4) - a_4 \omega_4 \dot{\psi}_4 \cos(\omega_4 t + \psi_4) \tag{8d}$$

Substituting $x, \dot{x}, \ddot{x}, \varphi, \dot{\varphi}, \ddot{\varphi}, u, \dot{u}, \ddot{u}, v, \dot{v}$ and $\ddot{v}$ from equations (5a) – (8d) into equations (1) – (4) and taking into account that $\cos\varphi = 1 - \dfrac{\varphi^2}{2!}$, $\sin\varphi = \varphi - \dfrac{\varphi^3}{3!}$ we obtain the following:

$$-\dot{a}_1 \omega_1 \sin(\omega_1 t + \psi_1) - a_1 \omega_1^2 \cos(\omega_1 t + \psi_1) - a_1 \omega_1 \dot{\psi}_1 \cos(\omega_1 t + \psi_1) - 2\varepsilon c_1 a_1 \omega_1 \sin(\omega_1 t + \psi_1)$$
$$+ a_1 \omega_1^2 \cos(\omega_1 t + \psi_1) + \alpha_1 a_1^2 \cos^2(\omega_1 t + \psi_1) + \varepsilon \alpha_2 a_1^3 \cos^3(\omega_1 t + \psi_1)$$
$$- (1 + a_1 \cos(\omega_1 t + \psi_1)) a_2^2 \omega_2^2 \sin^2(\omega_2 t + \psi_2) + \omega_2^2 \left[1 - (1 - \dfrac{\varphi^2}{2!}) \right]$$
$$= \varepsilon^2 f_1 \cos(\Omega_1 t) + \varepsilon \gamma_1 a_3^2 \cos^2(\omega_3 t + \psi_3) \tag{9a}$$

$$(1 + a_1 \cos(\omega_1 t + \psi_1))^2 \left(-\dot{a}_2 \omega_2 \sin(\omega_2 t + \psi_2) - a_2 \omega_2^2 \cos(\omega_2 t + \psi_2) - a_2 \omega_2 \dot{\psi}_2 \cos(\omega_2 t + \psi_2) \right)$$
$$- 2\varepsilon c_2 a_2 \omega_2 \sin(\omega_2 t + \psi_2) + 2(1 + a_1 \cos(\omega_1 t + \psi_1))(a_1 \omega_1 \sin(\omega_1 t + \psi_1))(a_2 \omega_2 \sin(\omega_2 t + \psi_2))$$
$$+ \omega_2^2 (1 + a_1 \cos(\omega_1 t + \psi_1))(\varphi - \dfrac{\varphi^3}{3!})$$
$$= \varepsilon^2 f_2 \cos(\Omega_2 t) + \varepsilon \gamma_2 a_4^2 \cos^2(\omega_4 t + \psi_4) \tag{9b}$$

$$-\dot{a}_3 \omega_3 \sin(\omega_3 t + \psi_3) - a_3 \omega_3 \dot{\psi}_3 \cos(\omega_3 t + \psi_3) - 2\varepsilon \mu_1 a_3 \omega_3 \sin(\omega_3 t + \psi_3)$$
$$= \varepsilon \lambda_1 a_3 a_1 \cos(\omega_3 t + \psi_3) \cos(\omega_1 t + \psi_1) \tag{9c}$$

$$-\dot{a}_4 \omega_4 \sin(\omega_4 t + \psi_4) - a_4 \omega_4 \dot{\psi}_4 \cos(\omega_4 t + \psi_4) - 2\varepsilon \mu_2 a_4 \omega_4 \sin(\omega_4 t + \psi_4)$$
$$= \varepsilon \lambda_2 a_4 a_2 \cos(\omega_4 t + \psi_4) \cos(\omega_2 t + \psi_2) \tag{9d}$$

Using equations (7a) – (7d) we get:

$$\dot{a}_1 = -\varepsilon c_1 a_1 + \varepsilon c_1 a_1 \cos\left(2\omega_1 t + 2\psi_1\right) + \frac{\alpha_1 a_1^2}{2\omega_1}\sin\left(\omega_1 t + \psi_1\right) + \frac{\alpha_1 a_1^2}{4\omega_1}\sin\left(3\omega_1 t + 3\psi_1\right)$$

$$+ \frac{\alpha_1 a_1^2}{4\omega_1}\sin\left(\omega_1 t + \psi_1\right) + \frac{\varepsilon\alpha_2 a_1^3}{4\omega_1}\sin\left(2\omega_1 t + 2\psi_1\right) + \frac{\varepsilon\alpha_2 a_1^3}{8\omega_1}\sin\left(4\omega_1 t + 4\psi_1\right)$$

$$- \frac{a_2^2\omega_2^2}{2\omega_1}\sin\left(\omega_1 t + \psi_1\right) + \frac{a_2^2\omega_2^2}{4\omega_1}\sin\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right) + \frac{a_2^2\omega_2^2}{4\omega_1}\sin\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right)$$

$$- \frac{a_1 a_2^2\omega_2^2}{4\omega_1}\sin\left(2\omega_1 t + 2\psi_1\right) + \frac{a_1 a_2^2\omega_2^2}{8\omega_1}\sin\left((2\omega_1 + 2\omega_2)t + (2\psi_1 + 2\psi_2)\right)$$

$$+ \frac{a_1 a_2^2\omega_2^2}{8\omega_1}\sin\left((2\omega_1 - 2\omega_2)t + (2\psi_1 - 2\psi_2)\right) + \frac{a_2^2\omega_2^2}{4\omega_1}\sin\left(\omega_1 t + \psi_1\right)$$

$$+ \frac{a_2^2\omega_2^2}{8\omega_1}\sin\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right) + \frac{a_2^2\omega_2^2}{8\omega_1}\sin\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right)$$

$$- \frac{\varepsilon^2}{2\omega_1}f_1\sin\left((\omega_1 + \Omega_1)t + \psi_1\right) - \frac{\varepsilon^2}{2\omega_1}f_1\sin\left((\omega_1 - \Omega_1)t + \psi_1\right) - \frac{\varepsilon\gamma_1 a_3^2}{2\omega_1}\sin\left(\omega_1 t + \psi_1\right)$$

$$- \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1}\sin\left((\omega_1 + 2\omega_3)t + (\psi_1 + 2\psi_3)\right) - \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1}\sin\left((\omega_1 - 2\omega_3)t + (\psi_1 - 2\psi_3)\right) \qquad (10a)$$

$$a_1\dot{\psi}_1 = -\varepsilon c_1 a_1 \sin\left(2\omega_1 t + 2\psi_1\right) + \frac{\alpha_1 a_1^2}{2\omega_1}\cos\left(\omega_1 t + \psi_1\right) + \frac{\alpha_1 a_1^2}{4\omega_1}\cos\left((-\omega_1)t + (-\psi_1)\right)$$

$$+ \frac{\alpha_1 a_1^2}{4\omega_1}\cos\left((3\omega_1)t + (3\psi_1)\right) + \frac{3\varepsilon\alpha_2 a_1^3}{8\omega_1} + \frac{\varepsilon\alpha_2 a_1^3}{2\omega_1}\cos\left(2\omega_1 t + 2\psi_1\right) + \frac{\varepsilon\alpha_2 a_1^3}{8\omega_1}\cos\left(4\omega_1 t + 4\psi_1\right)$$

$$- \frac{a_2^2\omega_2^2}{2\omega_1}\cos\left(\omega_1 t + \psi_1\right) + \frac{a_2^2\omega_2^2}{4\omega_1}\cos\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right)$$

$$+ \frac{a_2^2\omega_2^2}{4\omega_1}\cos\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right) - \frac{a_1 a_2^2\omega_2^2}{4\omega_1} + \frac{a_1 a_2^2\omega_2^2}{4\omega_1}\cos\left(2\omega_2 t + 2\psi_2\right)$$

$$- \frac{a_1 a_2^2\omega_2^2}{4\omega_1}\cos\left(2\omega_1 t + 2\psi_1\right) + \frac{a_1 a_2^2\omega_2^2}{8\omega_1}\cos\left((2\omega_1 - 2\omega_2)t + (2\psi_1 - 2\psi_2)\right)$$

$$+ \frac{a_1 a_2^2\omega_2^2}{8\omega_1}\cos\left((2\omega_1 + 2\omega_2)t + (2\psi_1 + 2\psi_2)\right) - \frac{\omega_2^2 a_2^2}{4\omega_1}\cos\left(\omega_1 t + \psi_1\right)$$

$$- \frac{\omega_2^2 a_2^2}{8\omega_1}\cos\left((2\omega_1 - 2\omega_2)t + (2\psi_1 - 2\psi_2)\right) + \frac{\omega_2^2 a_2^2}{8\omega_1}\cos\left((2\omega_1 + 2\omega_2)t + (2\psi_1 + 2\psi_2)\right)$$

$$- \frac{\varepsilon^2}{2\omega_1}f_1\cos\left((\Omega_1 - \omega_1)t - \psi_1\right) - \frac{\varepsilon^2}{2\omega_1}f_1\cos\left((\Omega_1 + \omega_1)t + \psi_1\right) + \frac{\varepsilon\gamma_1 a_3^2}{2\omega_1}\cos\left(\omega_1 t + \psi_1\right)$$

$$+ \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1}\cos\left((\omega_1 - 2\omega_3)t + (\psi_1 - 2\psi_3)\right) + \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1}\cos\left((\omega_1 + 2\omega_3)t + (\psi_1 + 2\psi_3)\right) \qquad (10b)$$

$$\dot{a}_2 = \frac{-a_2\omega_2}{2}\sin\left(2\omega_2 t + 2\psi_2\right) - \varepsilon c_2 a_2 + \varepsilon c_2 a_2 \cos\left(2\omega_2 t + 2\psi_2\right) + 2\varepsilon c_2 a_2 a_1 \cos\left(\omega_1 t + \psi_1\right)$$

$$-\varepsilon c_2 a_2 a_1 \cos\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right) - \varepsilon c_2 a_2 a_1 \cos\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right)$$

$$+ a_2 a_1 \omega_1 \sin\left(\omega_1 t + \psi_1\right) - \frac{a_2 a_1 \omega_1}{2}\sin\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right)$$

$$- \frac{a_2 a_1 \omega_1}{2}\sin\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right) - \frac{a_1^2 a_2 \omega_1}{2}\sin\left(2\omega_1 t + 2\psi_1\right)$$

$$+ \frac{a_1^2 a_2 \omega_1}{4}\sin\left((2\omega_1 + 2\omega_2)t + (2\psi_1 + 2\psi_2)\right) + \frac{a_1^2 a_2 \omega_1}{4}\sin\left((2\omega_1 - 2\omega_2)t + (2\psi_1 - 2\psi_2)\right)$$

$$+ \frac{\omega_2 a_2}{2}\sin\left(2\omega_2 t + 2\psi_2\right) - \frac{a_2^3 \omega_2}{24}\sin\left(2\omega_2 t + 2\psi_2\right) - \frac{a_2^3 \omega_2}{48}\sin\left(4\omega_2 t + 4\psi_2\right)$$

$$- \frac{a_1 a_2 \omega_2}{4}\sin\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right) + \frac{a_1 a_2 \omega_2}{4}\sin\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right)]$$

$$+ \frac{a_1 \omega_2 a_2^3}{48}\sin\left((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2)\right) + \frac{a_1 \omega_2 a_2^3}{96}\sin\left((\omega_1 + 4\omega_2)t + (\psi_1 + 4\psi_2)\right)$$

$$+ \frac{a_1 \omega_2 a_2^3}{96}\sin\left(\omega_1 t + \psi_1\right) - \frac{a_1 \omega_2 a_2^3}{48}\sin\left((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)\right) - \frac{a_1 \omega_2 a_2^3}{96}\sin\left(\omega_1 t + \psi_1\right)$$

$$- \frac{a_1 \omega_2 a_2^3}{96}\sin\left((\omega_1 - 4\omega_2)t + (\psi_1 - 4\psi_2)\right) + \frac{\varepsilon^2}{2\omega_2}f\sin\left((\omega_2 + \Omega_2)t + \psi_2\right)$$

$$+ \frac{\varepsilon^2}{2\omega_2}f\sin\left((\omega_2 - \Omega_2)t + \psi_2\right) - \frac{2a_1\varepsilon^2}{4\omega_2}f\sin\left((\omega_1 + \omega_2 + \Omega_2)t + (\psi_1 + \psi_2)\right)$$

$$+ \frac{2a_1\varepsilon^2}{4\omega_2}f\sin\left((\omega_1 - \omega_2 - \Omega_2)t + (\psi_1 - \psi_2)\right) - \frac{2a_1\varepsilon^2}{4\omega_2}f\sin\left((\omega_1 + \omega_2 - \Omega_2)t + (\psi_1 + \psi_2)\right)$$

$$- \frac{2a_1\varepsilon^2}{4\omega_2}f\sin\left((\omega_1 - \omega_2 + \Omega_2)t + (\psi_1 - \psi_2)\right) + \frac{\varepsilon\gamma_2 a_4^2}{2\omega_2}\sin\left(\omega_2 t + \psi_2\right)$$

$$+ \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2}\sin\left((\omega_2 + 2\omega_4)t + (\psi_2 + 2\psi_4)\right) + \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2}\sin\left((\omega_2 - 2\omega_4)t + (\psi_2 - 2\psi_4)\right)$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{2\omega_2}\sin\left((\omega_1 + \omega_2)t + (\psi_1 + \psi_2)\right) + \frac{\varepsilon\gamma_2 a_1 a_4^2}{2\omega_2}\sin\left((\omega_1 - \omega_2)t + (\psi_1 - \psi_2)\right)$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\sin\left((\omega_1 + \omega_2 + 2\omega_4)t + (\psi_1 + \psi_2 + 2\psi_4)\right) - \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\sin\left((\omega_1 + \omega_2 - 2\omega_4)t + (\psi_1 + \psi_2 - 2\psi_4)\right)$$

$$+ \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\sin\left((\omega_1 - \omega_2 + 2\omega_4)t + (\psi_1 - \psi_2 + 2\psi_4)\right)$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\sin\left((\omega_1 - \omega_2 - 2\omega_4)t + (\psi_1 - \psi_2 - 2\psi_4)\right)] \tag{11a}$$

$$a_2\dot{\psi}_2 = \frac{-a_2\omega_2}{2} - \frac{a_2\omega_2}{2}\cos(2\omega_2 t + 2\psi_2) - \varepsilon c_2 a_2 \sin(2\omega_2 t + 2\psi_2)$$

$$+ a_1\varepsilon c_2 a_2 \sin((2\omega_2 + \omega_1)t + (2\psi_2 + \psi_1)) + a_1\varepsilon c_2 a_2 \sin((2\omega_2 - \omega_1)t + (2\psi_2 - \psi_1))$$

$$+ \frac{a_2 a_1 \omega_1}{2}\cos((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)) - \frac{a_2 a_1 \omega_1}{2}\cos((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2))$$

$$- \frac{a_2 a_1^2 \omega_1}{4}\cos((2\omega_1 - 2\omega_2)t + (2\psi_1 - 2\psi_2)) - \frac{a_2 a_1^2 \omega_1}{4}\cos((2\omega_1 + 2\omega_2)t + (2\psi_1 + 2\psi_2))$$

$$+ \frac{\omega_2 a_2}{2} + \frac{\omega_2 a_2}{2}\cos(2\omega_2 t + 2\psi_2) - \frac{3\omega_2 a_2^3}{48} - \frac{\omega_2 a_2^3}{12}\cos(2\omega_2 t + 2\psi_2) - \frac{\omega_2 a_2^3}{48}\cos(4\omega_2 t + 4\psi_2)$$

$$- \frac{a_1 a_2 \omega_2}{2}\cos(\omega_1 t + \psi_1) - \frac{a_1 a_2 \omega_2}{4}\cos((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2))$$

$$- \frac{a_1 a_2 \omega_2}{4}\cos((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)) + \frac{2a_1 a_2^3 \omega_2}{48}\cos(\omega_1 t + \psi_1)$$

$$+ \frac{a_1 a_2^3 \omega_2}{24}\cos((\omega_1 - 2\omega_2)t + (\psi_1 - 2\psi_2)) + \frac{a_1 a_2^3 \omega_2}{24}\cos((\omega_1 + 2\omega_2)t + (\psi_1 + 2\psi_2))]$$

$$+ \frac{a_1 a_2^3 \omega_2}{96}\cos((\omega_1 - 4\omega_2)t + (\psi_1 - 4\psi_2)) + \frac{a_1 a_2^3 \omega_2}{96}\cos((\omega_1 + 4\omega_2)t + (\psi_1 + 4\psi_2))$$

$$+ \frac{\varepsilon^2}{2\omega_2}f_2\cos((\omega_2 - \Omega_2)t + \psi_2) + \frac{\varepsilon^2}{2\omega_2}f_2\cos((\omega_2 + \Omega_2)t + \psi_2)$$

$$- \frac{2a_1\varepsilon^2}{4\omega_2}f_2\cos((\omega_1 - \omega_2 - \Omega_2)t + (\psi_1 - \psi_2) - \frac{2a_1\varepsilon^2}{4\omega_2}f_2\cos((\omega_1 - \omega_2 + \Omega_2)t + (\psi_1 - \psi_2))$$

$$- \frac{2a_1\varepsilon^2}{4\omega_2}f_2\cos((\omega_1 + \omega_2 - \Omega_2)t + (\psi_1 + \psi_2) - \frac{2a_1\varepsilon^2}{4\omega_2}f_2\cos((\omega_1 + \omega_2 + \Omega_2)t + (\psi_1 + \psi_2)]$$

$$+ \frac{\varepsilon\gamma_2 a_4^2}{2\omega_2}\cos(\omega_2 t + \psi_2) + \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2}\cos((\omega_2 - 2\omega_4)t + (\psi_2 - 2\psi_4))$$

$$+ \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2}\cos((\omega_2 + 2\omega_4)t + (\psi_2 + 2\psi_4)) - \frac{\varepsilon\gamma_2 a_1 a_4^2}{2\omega_2}\cos((\omega_1 - \omega_2)t + (\psi_1 - \psi_2))$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{2\omega_2}\cos((\omega_1 + \omega_2)t + (\psi_1 + \psi_2)) - \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\cos((\omega_1 - (\omega_2 - 2\omega_4))t + \psi_1 - (\psi_2 - 2\psi_4))$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\cos((\omega_1 + (\omega_2 - 2\omega_4))t + \psi_1 + (\psi_2 - 2\psi_4))$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\cos((\omega_1 - (\omega_2 + 2\omega_4))t + \psi_1 - (\psi_2 + 2\psi_4))$$

$$- \frac{\varepsilon\gamma_2 a_1 a_4^2}{4\omega_2}\cos((\omega_1 + (\omega_2 + 2\omega_4))t + \psi_1 + (\psi_2 + 2\psi_4)) \tag{11b}$$

$$\dot{a}_3 = -\varepsilon\mu_1 a_3 + \varepsilon\mu_1 a_3 \cos\left(2\omega_3 t + 2\psi_3\right) + \frac{\varepsilon\lambda_1 a_3 a_1}{4\omega_3} \sin\left(\left(2\omega_3 + \omega_1\right)t + \left(2\psi_3 + \psi_1\right)\right)$$

$$+ \frac{\varepsilon\lambda_1 a_3 a_1}{4\omega_3} \sin\left(\left(2\omega_3 - \omega_1\right)t + \left(2\psi_3 - \psi_1\right)\right) \tag{12a}$$

$$a_3\dot{\psi}_3 = -\varepsilon\mu_1 a_3 \sin\left(2\omega_3 t + 2\psi_3\right) - \frac{\varepsilon\lambda_1 a_3 a_1}{2\omega_3} \cos\left(\omega_1 t + \psi_1\right)$$

$$- \frac{\varepsilon\lambda_1 a_3 a_1}{4\omega_3} \cos\left(\left(\omega_1 - 2\omega_3\right)t + \left(\psi_1 - 2\psi_3\right)\right) - \frac{\varepsilon\lambda_1 a_3 a_1}{4\omega_3} \cos\left(\left(\omega_1 + 2\omega_3\right)t + \left(\psi_1 + 2\psi_3\right)\right) \tag{12b}$$

$$\dot{a}_4 = -\varepsilon\mu_2 a_4 + \varepsilon\mu_2 a_4 \cos\left(2\omega_4 t + 2\psi_4\right) - \frac{\varepsilon\lambda_2 a_4 a_2}{4\omega_4} \sin\left(\left(2\omega_4 + \omega_2\right)t + \left(2\psi_4 + \psi_2\right)\right)$$

$$- \frac{\varepsilon\lambda_2 a_4 a_2}{4\omega_4} \sin\left(\left(2\omega_4 - \omega_2\right)t + \left(2\psi_4 - \psi_2\right)\right)] \tag{13a}$$

$$a_4\dot{\psi}_4 = -\varepsilon\mu_1 a_4 \sin\left(2\omega_4 t + 2\psi_4\right) - \frac{\varepsilon\lambda_2 a_4 a_2}{2\omega_4} \cos\left(\omega_2 t + \psi_2\right)$$

$$- \frac{\varepsilon\lambda_2 a_4 a_2}{2\omega_4} [\cos\left(\left(\omega_2 - 2\omega_4\right)t + \left(\psi_2 - 2\psi_4\right)\right) - \frac{\varepsilon\lambda_2 a_4 a_2}{2\omega_4} \cos\left(\left(\omega_2 + 2\omega_4\right)t + \left(\psi_2 + 2\psi_4\right)\right)] \tag{13b}$$

We can get the averaging equations conforming to simultaneous primary and internal 1:2 resonance by presenting the detuning parameters $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ according to $\Omega_1 = \omega_1 + \varepsilon\sigma_1, \Omega_2 = \omega_2 + \varepsilon\sigma_2, \omega_3 = \frac{1}{2}\omega_1 + \varepsilon\sigma_3, \omega_4 = \frac{1}{2}\omega_2 + \varepsilon\sigma_4$, keeping only the constant terms and slowly changing parts in equations (10a) – (13b). So, we'll have

$$\dot{a}_1 = \frac{-\varepsilon^2}{2\omega_1} f_1 \sin\theta_1 - \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1} \sin(\theta_3) - \varepsilon c_1 a_1 \tag{14a}$$

$$a_1\dot{\psi}_1 = \frac{3\varepsilon\alpha_2 a_1^3}{8\omega_1} - \frac{a_1 a_2^2 \omega_2^2}{4\omega_1} - \frac{\varepsilon^2}{2\omega_1} f_1 \cos\theta_1 + \frac{\varepsilon\gamma_1 a_3^2}{4\omega_1} \cos(\theta_3) \tag{14b}$$

$$\dot{a}_2 = \frac{\varepsilon^2}{2\omega_2} f_2 \sin\theta_2 - \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2} \sin(\theta_4) - \varepsilon c_2 a_2 \tag{15a}$$

$$a_2\dot{\psi}_2 = \frac{-3\omega_2 a_2^3}{48} - \frac{\varepsilon^2}{2\omega_2} f_2 \cos\theta_2 + \frac{\varepsilon\gamma_2 a_4^2}{4\omega_2} \cos(\theta_4) \tag{15b}$$

$$\dot{a}_3 = -\varepsilon\mu_1 a_3 - \frac{\varepsilon\lambda_1 a_3 a_1}{2\omega_1 + 4\varepsilon\sigma_3} \sin(\theta_3) \tag{16a}$$

$$a_3 \dot{\psi}_3 = -\frac{\varepsilon \lambda_1 a_3 a_1}{2\omega_1 + 4\varepsilon\sigma_3} \cos(\theta_3) \tag{16b}$$

$$\dot{a}_4 = -\varepsilon \mu_2 a_4 + \frac{\varepsilon \lambda_2 a_4 a_2}{2\omega_2 + 4\varepsilon\sigma_4} \sin(\theta_4) \tag{17a}$$

$$a_4 \dot{\psi}_4 = -\frac{\varepsilon \lambda_2 a_4 a_2}{2\omega_2 + 4\varepsilon\sigma_4} \cos(\theta_4) \tag{17b}$$

Where $\theta_1 = \psi_1 - \sigma_1 T$, $\theta_2 = \psi_2 - \sigma_2 T$, $\theta_3 = \psi_1 - 2\sigma_3 T_1 - 2\psi_3$, $\theta_4 = \psi_2 - 2\sigma_4 T_1 - 2\psi_4$

4 Stability Analysis

The stability for this system in equations (1)–(4) is checked at our selective case of resonance coincides to the invariant points of equations (14a) – (17b), that will be gotten by putting $\dot{a}_n = \dot{\theta}_n = 0$.

That is,

$$0 = \frac{-\varepsilon^2}{2\omega_1} f_1 \sin\theta_1 - \frac{\varepsilon \gamma_1 a_3^2}{4\omega_1} \sin(\theta_3) - \varepsilon c_1 a_1 \tag{18a}$$

$$a_1 \sigma_1 = \left(\frac{3\varepsilon \alpha_2 a_1^3}{8\omega_1} - \frac{a_1 a_2^2 \omega_2^2}{4\omega_1} \right) - \frac{\varepsilon^2}{2\omega_1} f_1 \cos\theta_1 + \frac{\varepsilon \gamma_1 a_3^2}{4\omega_1} \cos(\theta_3) \tag{18b}$$

$$0 = \frac{\varepsilon^2}{2\omega_2} f_2 \sin\theta_2 - \frac{\varepsilon \gamma_2 a_4^2}{4\omega_2} \sin(\theta_4) - \varepsilon c_2 a_2 \tag{19a}$$

$$a_2 \sigma_2 = \frac{-3\omega_2 a_2^3}{48} - \frac{\varepsilon^2}{2\omega_2} f_2 \cos\theta_2 + \frac{\varepsilon \gamma_2 a_4^2}{4\omega_2} \cos(\theta_4) \tag{19b}$$

$$0 = -\varepsilon \mu_1 a_3 - \frac{\varepsilon \lambda_1 a_3 a_1}{2\omega_1 + 4\varepsilon\sigma_3} \sin(\theta_3) \tag{20a}$$

$$a_3 \sigma_3 = \frac{1}{2} a_3 \sigma_1 + \frac{\varepsilon \lambda_1 a_3 a_1}{2\omega_1 + 4\varepsilon\sigma_3} \cos(\theta_3) \tag{20b}$$

$$0 = -\varepsilon \mu_2 a_4 + \frac{\varepsilon \lambda_2 a_4 a_2}{2\omega_2 + 4\varepsilon\sigma_4} \sin(\theta_4) \tag{21a}$$

$$a_4 \sigma_4 = \frac{1}{2} a_4 \sigma_2 + \frac{\varepsilon \lambda_2 a_4 a_2}{2\omega_2 + 4\varepsilon\sigma_4} \cos(\theta_4) \tag{21b}$$

From Eqs (20a) to (21b) we get:

$$\sin(\theta_3) = \frac{4\mu_1 \omega_3}{\lambda_1 a_1} \tag{22a}$$

$$\cos(\theta_3) = \frac{4\omega_3 \left(\sigma_3 - \frac{1}{2}\sigma_1 \right)}{\varepsilon \lambda_1 a_1} \tag{22b}$$

$$\sin(\theta_4) = \frac{4\mu_2 \omega_4}{\lambda_2 a_2} \tag{23a}$$

$$\cos(\theta_4) = \frac{4\omega_4 \left(\sigma_4 - \frac{1}{2}\sigma_2 \right)}{\varepsilon \lambda_2 a_2} \tag{23b}$$

For the practical case, $a_n \neq 0$, substituting by equations (22a) to (23b) and squaring equations (18a), (18b), then taking the squared results in a computation process, likewise equations (19a), (19b), equations (20a), (20b) and equations (21a), (21b) deduce the following frequency response equations:

$$\frac{\varepsilon^4}{4\omega_1^2} f_1^2 = \varepsilon^2 c_1^2 a_1^2 + \frac{\varepsilon^2 \gamma_1^2 a_3^4}{16\omega_1^2} + \left(-a_1 \sigma_1 + \frac{3\varepsilon\alpha_2 a_1^3 - 2a_1 a_2^2 \omega_2^2}{8\omega_1} \right)^2 + \frac{\varepsilon^2 c_1 a_1 \gamma_1 a_3^2}{2\omega_1} \sin(\theta_3)$$

$$+ \frac{\varepsilon \gamma_1 a_3^2}{2\omega_1} \left(-a_1 \sigma_1 + \frac{3\varepsilon\alpha_2 a_1^3 - 2a_1 a_2^2 \omega_2^2}{8\omega_1} \right) \cos(\theta_3) \tag{24a}$$

$$\frac{\varepsilon^4}{4\omega_2^2} f_2^2 = \frac{\varepsilon^2 \gamma_2^2 a_4^4}{16\omega_2^2} + \varepsilon^2 c_2^2 a_2^2 + \left(-a_2 \sigma_2 - \frac{3\omega_2 a_2^3}{48} \right)^2 + \frac{\varepsilon^2 c_2 a_2 \gamma_2 a_4^2}{2\omega_2} \sin(\theta_4)$$

$$+ \frac{\varepsilon \gamma_2 a_4^2}{2\omega_2} \left(-a_2 \sigma_2 - \frac{3\omega_2 a_2^3}{48} \right) \cos(\theta_4) \tag{24b}$$

$$\left(\frac{\varepsilon \lambda_1 a_1}{2\omega_1 + 4\varepsilon\sigma_3} \right)^2 = \varepsilon^2 \mu_1^2 + (\sigma_3 - \frac{1}{2}\sigma_1)^2 \tag{24c}$$

$$\left(\frac{\varepsilon \lambda_2 a_2}{2\omega_2 + 4\varepsilon\sigma_4} \right)^2 = \varepsilon^2 \mu_2^2 + (\sigma_4 - \frac{1}{2}\sigma_2)^2 \tag{24d}$$

5 Nonlinear Solution

The stability for this system was specified by examining the eigenvalues of the right-hand sides of equations (14a) – (17b) which represent as the Jacobian matrix. The equipoise solution is approximately stable as long as the corresponding eigenvalue's real part is negative. If not, the corresponding result is unstable. To deduce the stability criteria, we just need to check the demeanor of insignificant perturbations from the stabilised-case solutions a_{n0} and θ_{n0}.

So, we suppose the following:

$$a_n = a_{n0} + a_{n1}, \qquad \theta_n = \theta_{n0} + \theta_{n1},$$
$$\dot{a}_n = \dot{a}_{n1}, \qquad \dot{\theta}_n = \dot{\theta}_{n1}, \tag{25}$$

Where a_{n0}, θ_{n0} are the solutions of equations (14a) – (17b) and a_{n1}, θ_{n1} are known as perturbations which are presumed to be very small compared with a_{n0}, θ_{n0}. Replacing equation (25) into equations (14a) – (17b) and conserving the linear expressions in a_{n1}, θ_{n1} only. We obtain;

$$\dot{a}_{11} = -\varepsilon c_1 a_{11} - \frac{\varepsilon^2}{2\omega_1} f_1\left(\cos\theta_{10}\right)\theta_{11} - \frac{\varepsilon\gamma_1(2a_{30})}{4\omega_1}\left(\sin\theta_{30}\right)a_{31} - \frac{\varepsilon\gamma_1 a_{30}^2}{4\omega_1}\left(\cos\theta_{30}\right)\theta_{31}$$

$$(26a) \quad \dot{\theta}_{11} = \left(-\frac{\sigma_1}{a_{10}} + \frac{9\varepsilon\alpha_2 a_{10}}{8\omega_1} - \frac{a_{20}^2\omega_2^2}{4a_{10}\omega_1}\right)a_{11} - \frac{\varepsilon^2}{2a_{10}\omega_1} f_1(\sin\theta_{10})\theta_{11} - \frac{a_{20}}{2\omega_1}a_{21}$$

$$- \frac{\varepsilon\gamma_1 a_{30}}{2a_{10}\omega_1}\left(\cos\theta_{30}\right)a_{31} + \frac{\varepsilon\gamma_1 a_{30}^2}{4a_{10}\omega_1}\left(\sin\theta_{30}\right)\theta_{31} \tag{26b}$$

$$\dot{a}_{21} = -\varepsilon c_2 a_{21} - \frac{\varepsilon^2}{2\omega_2} f_2\left(\cos\theta_{20}\right)\theta_{21} - \frac{\varepsilon\gamma_2 a_{40}}{2\omega_2}\left(\sin\theta_{40}\right)a_{41} - \frac{\varepsilon\gamma_2 a_{40}^2}{4\omega_2}\left(\cos\theta_{40}\right)\theta_{41} \tag{27a}$$

$$\dot{\theta}_{21} = \left(-\frac{\sigma_2}{a_{20}} - \frac{9\omega_2}{48}\right)a_{21} - \frac{\varepsilon^2}{2a_{20}\omega_2} f_2(\sin\theta_{20})\theta_{21} - \frac{\varepsilon\gamma_2 a_{40}}{2a_{20}\omega_2}\left(\cos\theta_{40}\right)a_{41} + \frac{\varepsilon\gamma_2 a_{40}^2}{4a_{20}\omega_2}\left(\sin\theta_{40}\right)\theta_{41} \tag{27b}$$

$$\dot{a}_{31} = \frac{\varepsilon\lambda_1 a_{30}}{4\omega_3}\sin\theta_{30}\,a_{11} + \left(-\varepsilon\mu_1 + \frac{\varepsilon\lambda_1 a_{10}}{4\omega_3}\sin\theta_{30}\right)a_{31} + \frac{\varepsilon\lambda_1 a_{30}a_{10}}{4\omega_3}\left(\cos\theta_{30}\right)\theta_{31} \tag{28a}$$

$$\dot{\theta}_{31} = -\frac{\varepsilon\lambda_1 a_{30}}{4a_{30}\omega_3}\left(\cos\theta_{30}\right)a_{11} + \left(-\frac{\sigma_3}{a_{30}} - \frac{\varepsilon\lambda_1 a_{10}}{4a_{30}\omega_3}\cos\theta_{30}\right)a_{31} + \frac{\varepsilon\lambda_1 a_{10}}{4\omega_3}\left(\sin\theta_{30}\right)\theta_{31} \tag{28b}$$

$$\dot{a}_{41} = \frac{\varepsilon\lambda_2 a_{40}}{4\omega_4}\left(\sin\theta_{40}\right)a_{21} + \left(-\varepsilon\mu_2 + \frac{\varepsilon\lambda_2 a_{20}}{4\omega_4}\sin\theta_{40}\right)a_{41} + \frac{\varepsilon\lambda_2 a_{40}a_{40}}{4\omega_4}\left(\cos\theta_{40}\right)\theta_{41} \tag{29a}$$

$$\dot{\theta}_{41} = -\frac{\varepsilon\lambda_2 a_{40}}{4a_{40}\omega_4}\left(\cos\theta_{40}\right)a_{21} + \left(-\frac{\sigma_4}{a_{40}} - \frac{\varepsilon\lambda_2 a_{20}}{4a_{40}\omega_4}\cos\theta_{40}\right)a_{41} + \frac{\varepsilon\lambda_2 a_{20}}{4\omega_4}\left(\sin\theta_{40}\right)\theta_{41} \tag{29b}$$

The previous equations could be presented in the matrix form:

$$\begin{bmatrix} \dot{a}_{11} & \dot{\vartheta}_{11} & \dot{a}_{21} & \dot{\vartheta}_{21} & \dot{a}_{31} & \dot{\vartheta}_{31} & \dot{a}_{41} & \dot{\vartheta}_{41} \end{bmatrix}^T = [J] \begin{bmatrix} a_{11} & \vartheta_{11} & a_{21} & \vartheta_{21} & a_{31} & \vartheta_{31} & a_{41} & \vartheta_{41} \end{bmatrix}^T \qquad (30)$$

Where $[J]$ is the Jacobin of the right hand sides of equations (26a) – (29b). Now let us put the eigenvalues of the above system of equations $[J]$ in the following form:

$$\xi^8 + S_1\xi^7 + S_2\xi^6 + S_3\xi^5 + S_4\xi^4 + S_5\xi^3 + S_6\xi^2 + S_7\xi + S_8 = 0 \qquad (31)$$

If and only if the real part of the eigenvalue, which obtained from Eigen equation (31), is negative, then the solution is stable; otherwise, the solution is going to be unstable. The necessary and sufficient conditions for all the roots of Eq. (31) will be calculated corresponding to the Routh-Hurwitz criterion.

$$D = \begin{vmatrix} S_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_3 & S_2 & S_1 & 1 & 0 & 0 & 0 & 0 \\ S_5 & S_4 & S_3 & S_2 & S_1 & 1 & 0 & 0 \\ S_7 & S_6 & S_5 & S_4 & S_3 & S_2 & S_1 & 1 \\ 0 & S_8 & S_7 & S_6 & S_5 & S_4 & S_3 & S_2 \\ 0 & 0 & 0 & S_8 & S_7 & S_6 & S_5 & S_4 \\ 0 & 0 & 0 & 0 & 0 & S_8 & S_7 & S_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_8 \end{vmatrix} \qquad (32)$$

6 Numerical Solutions

The basic system with NSC controllers which expressed in the differential equations form (1-4) was solved by applying Rung–Kutta 4th order method numerically. The emulation results are attained by utilising MATLAB 7.14 (R2013a).
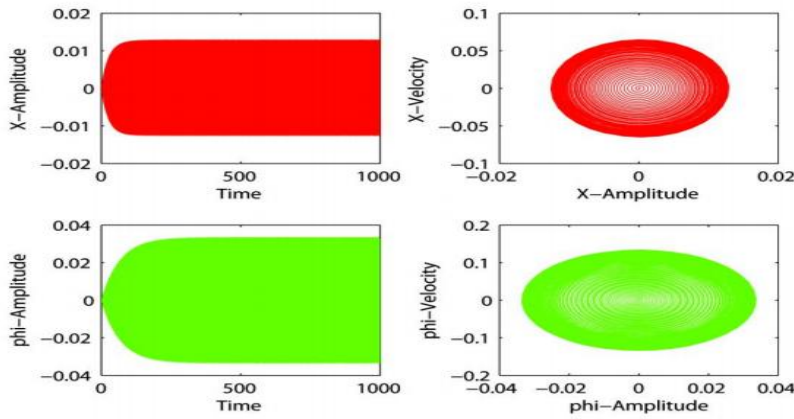
Fig. 2. The fundamental system (x , φ) without controllers
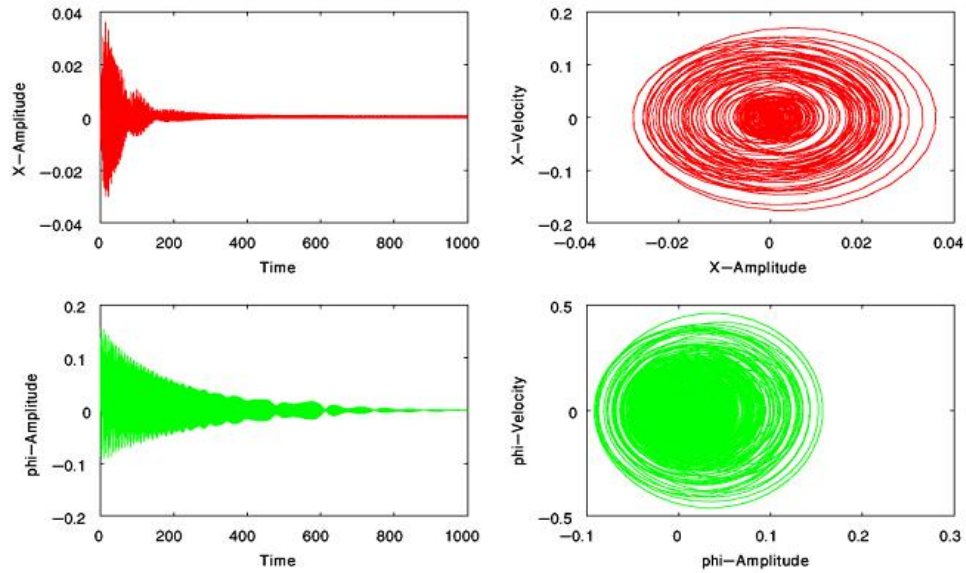
Fig. 3. Response of the fundamental system with NSC controllers
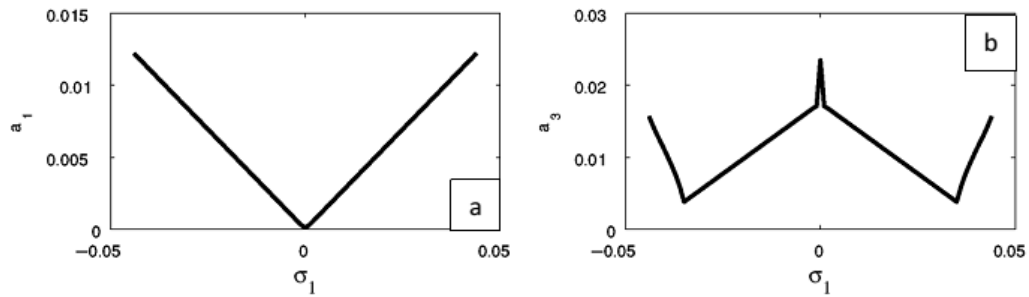
Fig. 4. Frequency response curves of: (a) the fundamental system (b) the NSC

Table 1. Frequency response curve with the detuning parameter (σ_1)

Parameters	Effect	Figures
The external forcing f_1	While the value of f_1 was increasing, the amplitude of the fundamental system and the NSC increased.	**Fig.(5)**
The damping coefficients c_1 The natural frequencies ω_1 The control signal gains γ_1	It is noticed that the increasing in the values of c_1, ω_1 and γ_1 led to the decreasing in the magnitude of amplitude of the fundamental system and the NSC	**Fig.(6,7,8)**
The detuning parameter σ_3	In the case of increasing the value of σ_3, the amplitude of the fundamental system and the NSC are shifted to the right.	**Fig.(9)**
The damping coefficients μ_1	When the value of μ_1 increased, the magnitude of amplitude of the fundamental system and the NSC were decreasing.	**Fig.(10)**

14
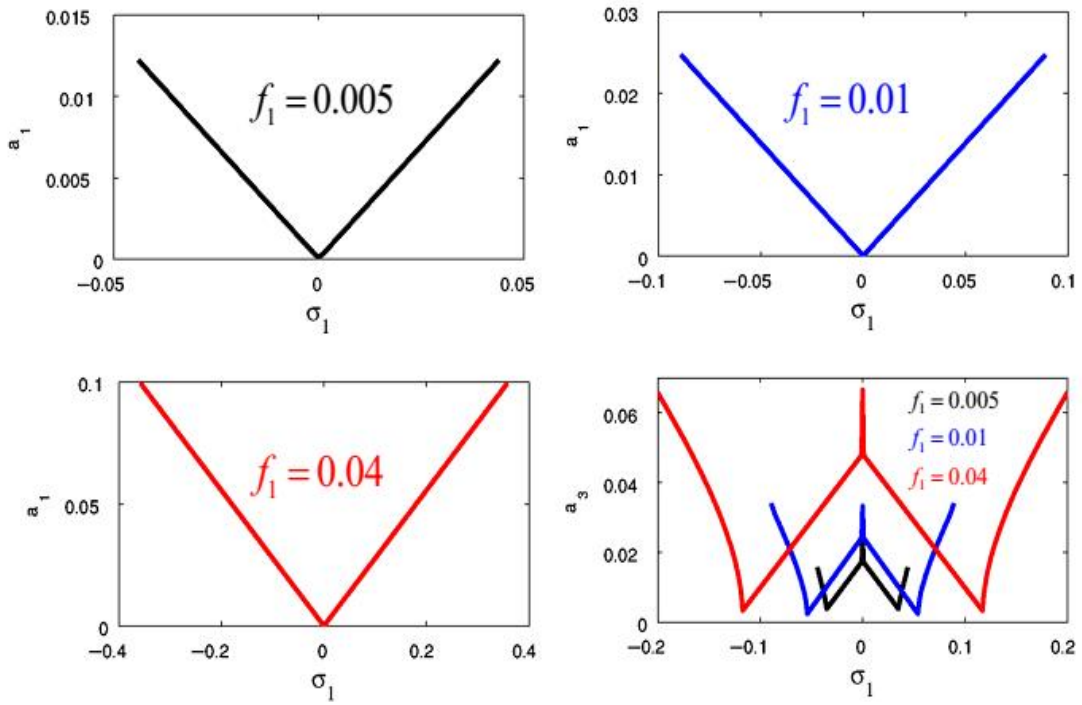
Fig. 5. Influence of f_1 (the external force) on the fundamental system and the NSC
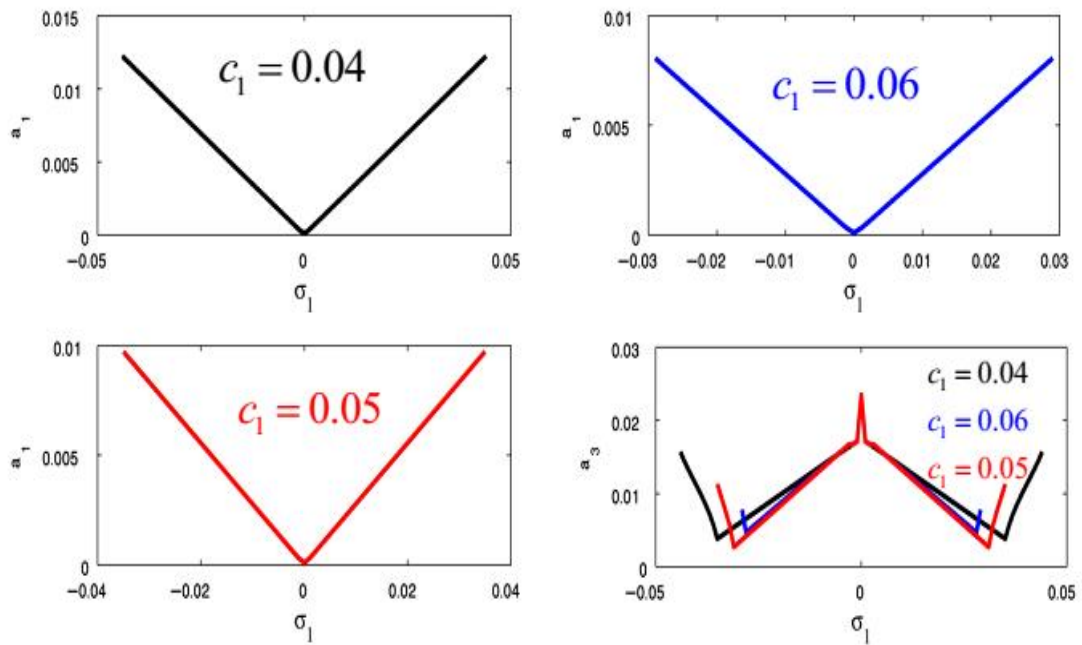
Fig. 6. Influence of c_1 (the damping coefficient) on the fundamental system and the NSC
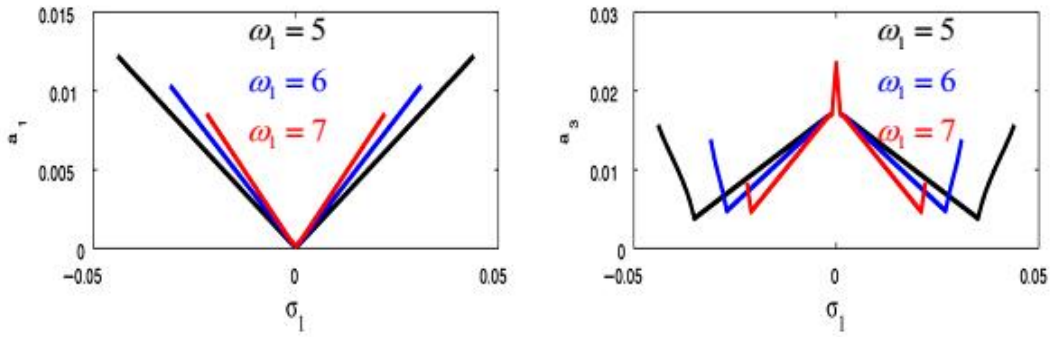
Fig. 7. Influence of ω_1 (the natural frequency) on the fundamental system and the NSC
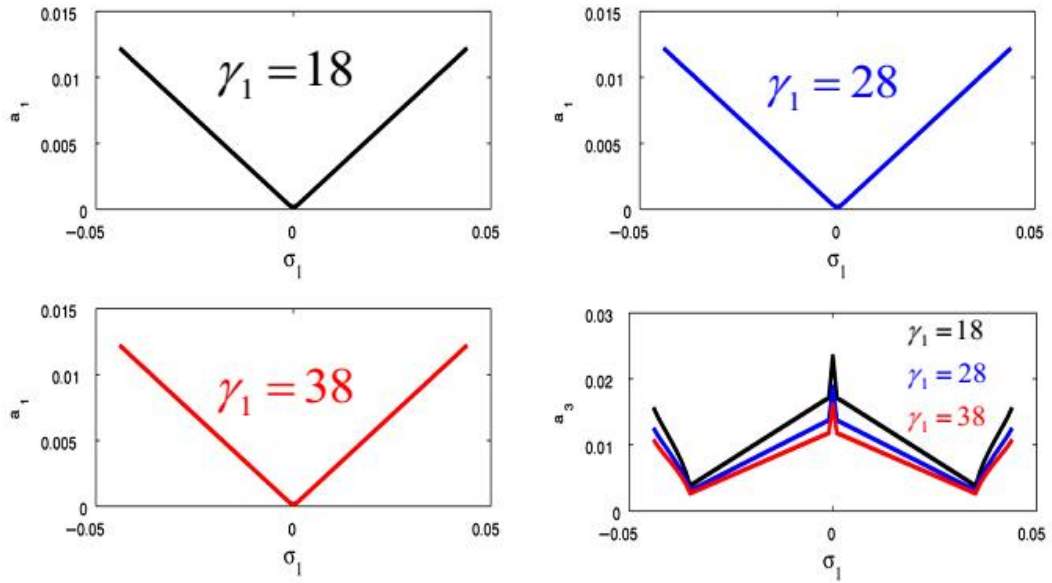
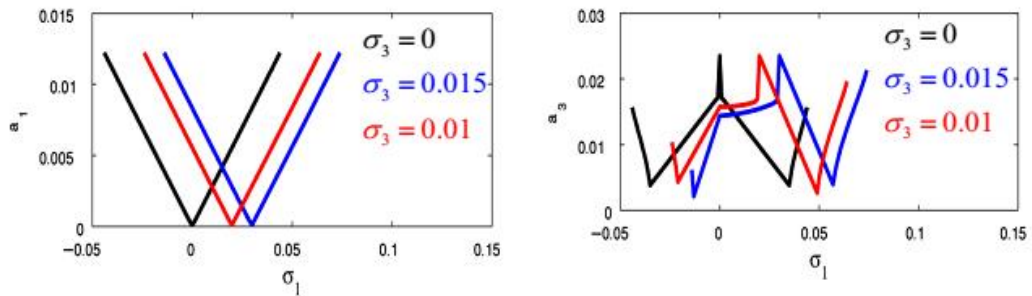**Fig. 8. Influence of γ_1 (the control signal gain) on the fundamental system and the NSC**

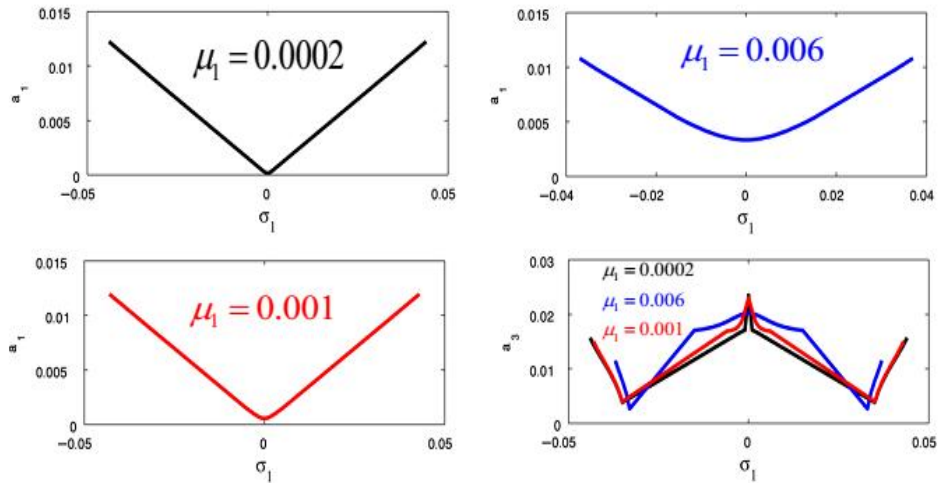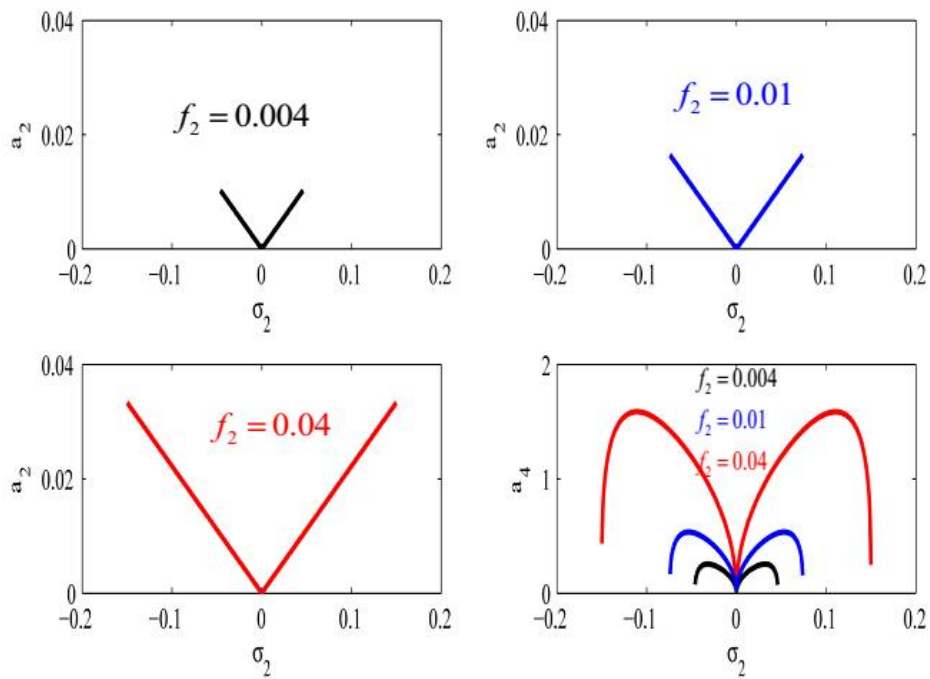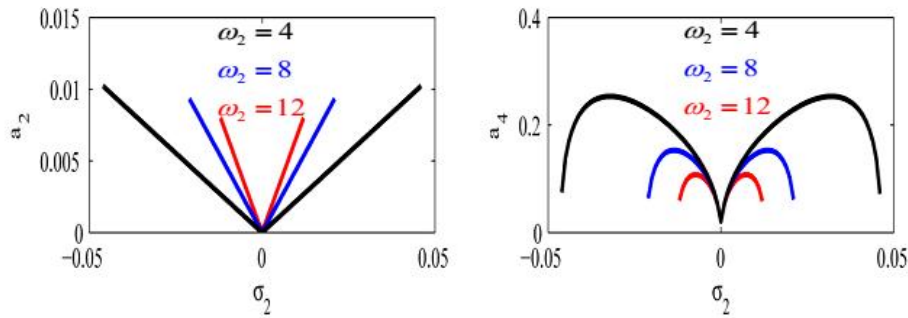**Fig. 9. Influence of σ_3 (the detuning parameter) on the fundamental system and the NSC**

Fig. 10. Influence of μ_1 (the damping coefficient) on the fundamental system and the NSC

Fig. 11. Influence of f_2 (the external excitation force) on the fundamental system and the NSC

Table 2. Frequency response curve with the detuning parameter (σ_2)

Parameters	Effect	Figures
The external forcing f_2	While we were increasing the values of f_2 the amplitude of the fundamental system and the NSC increased	Fig. (11)

Parameters	Effect	Figures
The natural frequency ω_2 The damping coefficient c_2	In the case of increasing the values of ω_2 and c_2, the magnitude of amplitude of the fundamental system and the NSC are decreased.	**Fig. (12,16)**
The control signal gain γ_2 The damping coefficient μ_2	When the values of γ_2 and μ_2 were increasing, the amplitude of the NSC decreased	**Fig(13,17)**
The detuning parameter σ_4	In the case of increasing the value of σ_4 the amplitude of the fundamental system and the NSC are shifted to the right.	**Fig(14)**
The Feedback signal gain λ_2	If we increase the value of λ_2, we notice that the amplitude of the system decreased while it increased in the NSC.	**Fig(15)**

Fig. 12. Influence of ω_2 (the natural frequency) on the fundamental system and the NSC
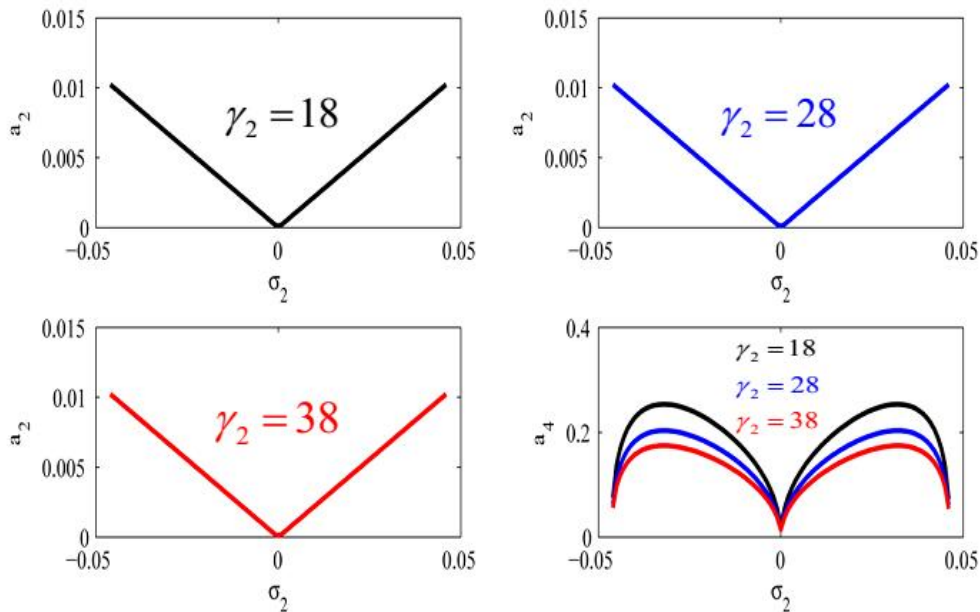

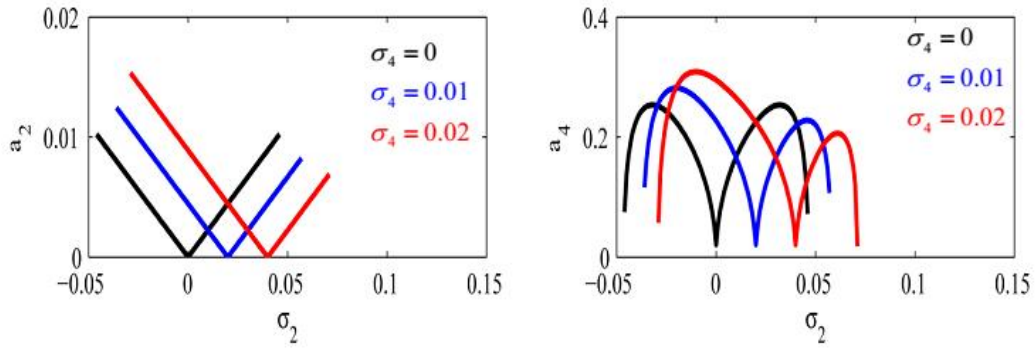**Fig. 13. Influence of γ_2 (the control signal gain) on the fundamental system and the NSC**

Fig. 14. Influence of σ_4 (the detuning parameter) on the fundamental system and the NSC
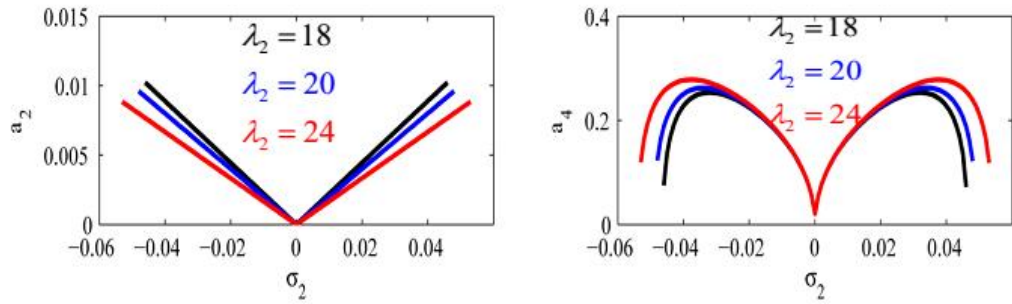

Fig. 15. Influence of λ_2 (the feedback signal gain) on the fundamental system and the NSC
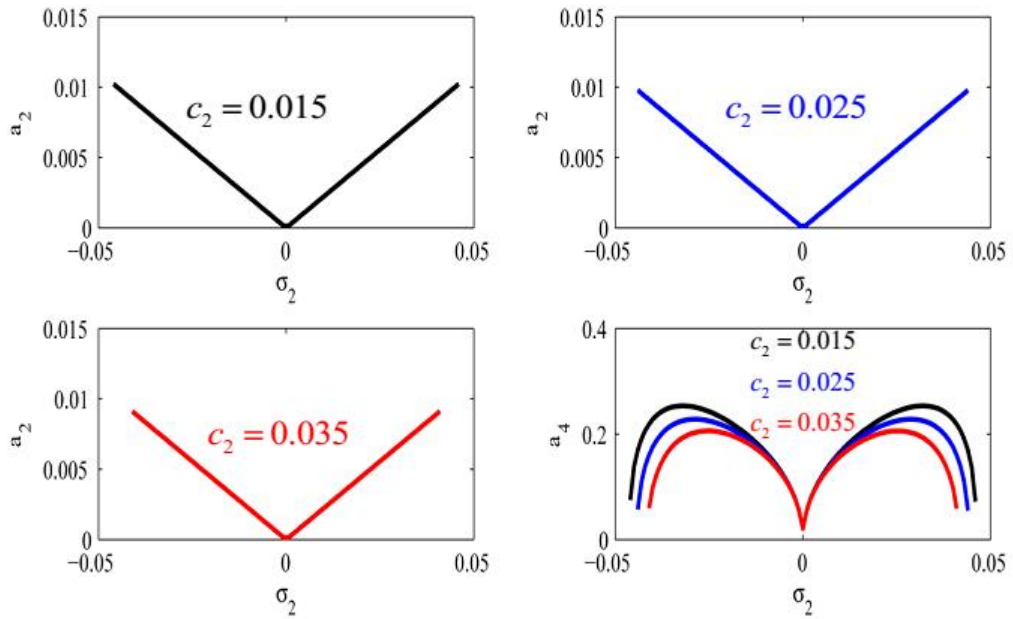

Fig. 16. Influence of c_2 (the damping coefficient) on the fundamental system and the NSC
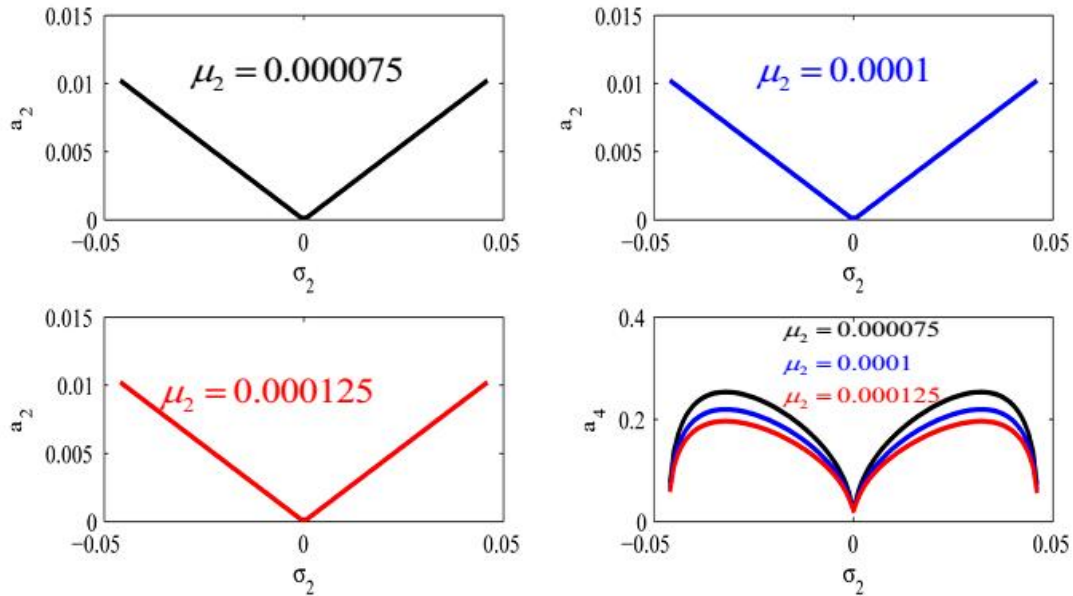
Fig. 17. Influence of μ_2 (the damping coefficient) on the fundamental system and the NSC

7 Conclusions

The NSC controllers for suppressing of the oscillations of the non-linear spring pendulum have been studied. We represent the problem by non-linear ordinary differential equation system with four-degree-of-freedom. The averaging method is used for the mathematical analysis. for the case of primary resonance in the presence of 1:2 internal resonances, the frequency response equations have been derived. The system's stability has been discussed by applying the frequency response equations and the phase plane technique. It is worth to notice that the steady-state amplitudes of the spring pendulum with NSC controllers were reduced to about 97.8% in both directions (x , φ) from its value without NSC controllers.

The influences of the diversified parameters of the system are surveyed numerically. This survey makes the frequency response curve with the detuning parameter (σ_1) is clear due to different parameters. And it was noticed that:

- The amplitude of the system was increasing when the value of f_1 increased.
- While the values of $c_1, \omega_1, \gamma_1 \ and \ \mu_1$ were increasing, the values of the amplitude of the system decreased.

Also, we studied the effectiveness of distinguished parameters on the frequency response curve with the detuning parameter (σ_2). The most obviousness features are:

- The amplitude of the system was direct proportional to the values of f_2 .
- The amplitude of the system was inverse proportional to the values of the following parameters $c_2, \omega_2, \gamma_2, \lambda_2 \ and \ \mu_2$

The effectiveness of the controller Ea (Ea= steady-state amplitudes of the system without controller/steady-state amplitudes with controller) is about 45.3 for (x) and 40.5 for (φ).

Competing Interests

Authors have declared that no competing interests exist.

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