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A Comparative Study for Solving Nonlinear Fractional Heat -Like Equations via Elzaki Transform

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, the Homotopy Perturbation Elzaki Transform Method (HPETM) and Homotopy Decomposition Method (HDM) are used to solve nonlinear fractional Heat - Like equations. Both methods are very efficient techniques and quite capable, practically for solving different kinds of linear and nonlinear fractional differential equations .The results reveal that the (HDM) has an advantage over the (HPETM) which is that it solves the nonlinear problems using only the inverse operator which is basically the fractional integral. Additionally there is no need to use any other inverse transform to find the components of the series solutions as in the case of HPETM. As a consequence the calculations involved in HDM are very simple and easy execution.

Keywords: Homotopy decomposition method; integral transforms; nonlinear Heat -Like equation; Elzaki transform.

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1 Introduction

Fractional calculus [1-3] is a generalization of differentiation and integration to non-integer orders. Many problems in physics and engineering are modulated in terms of fractional differential and integral equations, such as acoustics, diffusion, signal processing, electrochemistry, and many other physical phenomena. During the past few decades, a great deal of interest appears in fractional differential equations. The solutions of fractional equations [4-15] are investigated by many authors using powerful methods in obtaining exact and approximate solutions, Among these numerical methods, the Variational Iteration Method (VIM) [Biazar and Ghazvini (2007)], Adomian Decomposition Method (ADM) [Hashim, Noorani, Ahmed. Bakar. Ismail and Zakaria, (2006)] [16-17], and the Differential Transform Method (ADM) are the most popular ones that are used to solve differential and integral equations of integer and fractional order.

The Homotopy perturbation method (HPM) is proposed by He in 1999 [18-22]. This method is a coupling of traditional perturbation method and homotopy in topology. In recent years Homotopy perturbation method has been extensively introduced by numerous authors, and implemented to obtain exact and approximate analytical solutions to a wide range of both linear and nonlinear problems in science and engineering The Homotopy decomposition method (HDM) was recently proposed by [23-24] to solve the groundwater flow equation and the modified fractional KDV equation [25]. The Homotopy decomposition method [26] is actually the combination of perturbation method and Adomian Decomposition Method. Recently,Tarig M. Elzaki and Sailh M. Elzaki in [27-32], showed Elzaki transform, was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations.

In this paper, the main objective is to introduce a comparative study of nonlinear fractional Heat -Like equations by using the Homotopy Perturbation Elzaki Transform Method (HPETM) which is the coupling of the Elzaki transform and the HPM using He's polynomials. And the Homotopy Decomposition Method (HDM).

2 Fundamental Facts of Elzaki Transform

A new transform called the Elzaki transform defined for function of exponential order we consider functions in the set **A,** defined by:

$$A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{if } t \in (-1)^j \times [0,\infty) \tag{1}$$

For a given function in the set, the constant M must be finite number,k_1, k_2 may be finite or infinite. The Elzaki transform which is defined by the integral equation

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{\frac{-t}{v}}\,dt \qquad , t \geq 0, k_1 \leq v \leq k_2 \tag{2}$$

$$E[f(t), v] = T(v) = v\int_0^t f(t)e^{-\frac{t}{v}}dt, \quad v \in (k_1, k_2)$$

The following results can be obtained from the definition and simple calculations

1) $E[t^n] = n!\,v^{n+2}$
2) $E[f'(t)] = \frac{T(v)}{v} - vf(0)$
3) $E[f''(t)] = \frac{T(v)}{v^2} - f(0) - vf'(0)$
4) $E\left[f^{(n)}(t)\right] = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0).$

3 Fundamental Facts of the Fractional Calculus

Definition 1: A real function $f(x), x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in [0, \infty)$ and it is said to be in space C_μ^m if $f^{(m)} \in C_m, m \in \mathbb{N}$.

Definition 2: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_m$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-1)^{\alpha-1} f(t) dt, \alpha > 0\,, x > 0 \qquad (3)$$
$$J^\alpha f(x) = f(x)$$

Let's consider a some of properties for operator J^α (e.g., [1-3]):

If $f \in C_m$, $\mu \geq -1\,, \alpha\,, \beta \geq 0$ and $\gamma > -1$ then $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$, $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$ $\quad J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Lemma 1:

If $m - 1 < \alpha \leq m\,, m \in \mathbb{N}$ and $f \in C_m, \mu \geq -1$ then $D^\alpha J^\alpha f(x) = f(x)$ and,

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}\,, x > 0 \qquad (4)$$

Definition 3: (Partial Derivatives of Fractional order)

Assume now that $f(x)$ is a function of n variables x_i, $i = 1, \dots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of definition 3 we define partial derivative of order α for $f(x)$ respect to x_i

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_0^{x_i} (x_i - 1)^{m-\alpha-1} \partial_{x_i}^\alpha f(x_j)\Big|_{x_j=t} dt \qquad (5)$$

If it exists, where $\partial_{x_i}^\alpha$ is the usual partial derivative of integer order m.

Theorem 1:

If $T(v)$ is Elzaki transform of (t), one can take into consideration the Elzaki transform of the Riemann-Liouville derivative as follow:

$$T[D^\alpha f(t)] = v^{-\alpha}[T(v) - \sum_{k=1}^{n} v^{\alpha-k+2}\,[D^{\alpha-k} f(0)]] \quad ; -1 < n - 1 \leq \alpha < n \qquad (6)$$

Proof: Let us take Laplace transformation of $f'(t) = \frac{d}{dt} f(t)$

$$L[D^\alpha f(t)] = S^\alpha T(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(0)]$$
$$= s^\alpha T(s) - \sum_{k=0}^{n} s^{k-1}[D^{\alpha-k} f(0)] = s^\alpha T(s) - \sum_{k=1}^{n} s^{k-2}[D^{\alpha-k} f(0)]$$
$$= s^\alpha T(s) - \frac{1}{s^{-k+2}} \sum_{k=1}^{n} [D^{\alpha-k} f(0)] = s^\alpha T(s) - \sum_{k=0}^{n} \frac{1}{s^{\alpha-k+2-\alpha}} [D^{\alpha-k} f(0)]$$
$$= s^\alpha T(s) - \sum_{k=1}^{n} s^\alpha \frac{1}{s^{\alpha-k+2}} [D^{\alpha-k} f(0)]$$

$$L[D^{\alpha}f(t)] = s^{\alpha}\left[T(s) - \sum_{k=1}^{n}\left(\frac{1}{s}\right)^{\alpha-k+2}[D^{\alpha-k}f(0)]\right]$$

Therefore, when we substitute $\frac{1}{v}$ for s, we get the Elzaki transformation of fractional order of $f(t)$ as follows:

$$E[D^{\alpha}f(t)] = v^{-\alpha}[T(v) - \sum_{k=1}^{n} v^{\alpha-k+2}[D^{\alpha-k}f(0)]] \quad (7)$$

Definition 4:

The Elzaki transform of the Caputo fractional derivative by using Theorem 1 is defined as follows:

$$E[D_t^{\alpha}f(t)] = v^{-\alpha}E[f(t)] - \sum_{k=0}^{m-1} v^{\alpha-k+2}f^{(k)}(0) \quad , m-1 < \alpha < m \quad (8)$$

4 Basic Idea

4.1 Basic idea of HPETM

To illustrate the basic idea of this method, we consider a general form of nonlinear non homogeneous partial differential equation as the follow:

$$D_t^{\alpha}u(x,t) = L\big(u(x,t)\big) + N\big(u(x,t)\big) + f(x,t) \quad , \quad \alpha > 0 \quad (9)$$

with the following initial conditions

$$D_0^k u(x,0) = g_k, k = 0, \dots, n-1 \ , \ D_0^n u(x,0) = 0 \text{ and } n = [\alpha] \quad (10)$$

where D_t^{α} denotes without loss of generality the Caputo fractional derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Taking Elzaki transform on both sides of equation (9), to get:

$$E[D_t^{\alpha}u(x,t)] = E\big[L\big(u(x,t)\big)\big] + E\big[N\big(u(x,t)\big)\big] + E[f(x,t)] \quad (11)$$

Using the differentiation property of Elzaki transform and above initial conditions, we have:

$$E[u(x,t)] = v^{\alpha}E\big[L\big(u(x,t)\big)\big] + v^{\alpha}E\big[N\big(u(x,t)\big)\big] + g(x,t) \quad (12)$$

Operating with the Elzaki inverse on both sides of equation (12) gives:

$$u(x,t) = G(x,t) + E^{-1}\left[v^{\alpha}E\big[L\big(u(x,t)\big)\big] + v^{\alpha}E\big[N\big(u(x,t)\big)\big]\right] \quad (13)$$

where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial condition.

Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \quad (14)$$

And the nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (15)$$

where $H_n(u)$are He's polynomial and given by:

$$H_n(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i u_i(x,t))]_{p=0} \ , n = 0,1,2, \dots \quad (16)$$

Substituting equations. (15) and (14) in equation (13) we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left[E^{-1}\left[v^\alpha E[L(\sum_{n=0}^{\infty} p^n u_n(x,t))] + v^\alpha E[N(\sum_{n=0}^{\infty} p^n u_n(x,t))]\right]\right] \quad (17)$$

which is the coupling of the Elzaki transform and the homotopy perturbation method using He's polynomials and after Comparing the coefficient of like powers of p, we obtain the following approximations:

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = E^{-1}[v^\alpha E[L(u_0(x,t)) + H_0(u)]],$$

$$p^2 : u_2(x,t) = E^{-1}[v^\alpha E[L(u_1(x,t)) + H_1(u)]],$$

$$p^3 : u_3(x,t) = E^{-1}[v^\alpha E[L(u_2(x,t)) + H_2(u)]],$$

$$p^n : u_n(x,t) = E^{-1}[v^\alpha E[L(u_{n-1}(x,t)) + H_{n-1}(u)]], \quad (18)$$

Hence, the solution can be expressed in the form

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t)+u_1(x,t) + u_2(x,t) + \cdots \quad (19)$$

By virtue of (18) the solution (19) is converges very rapidly..

4.2 Basic idea of HDM

The method consists of first step to transform the fractional partial differential equation to the fractional partial integral equation which applying the inverse operator D_t^α to the both sides of equation (9), finally, solution u(x,t) can be written in the form:

$$u(x,t) = \sum_{j=1}^{n-1} \frac{g_j}{\Gamma(\alpha - j + 1)} t^j + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[L\left(u(x,\tau)\right) + N\left(u(x,\tau)\right) + f(x,\tau)\right]d\tau \quad (20)$$

Other side using the following

$$\sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha - j + 1)} t^{\alpha - j} = f(x,t) \text{or} \sum_{j=1}^{n-1} \frac{g_j}{\Gamma(\alpha - j + 1)} t^j = f(x,t)$$

we have

$$u(x,t) = T(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[L\left(u(x,\tau)\right) + N\left(u(x,\tau)\right) + f(x,\tau)\right]d\tau \quad (21)$$

In the method of homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p

$$u(x,t,p) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (22)$$

$$u(x,t) = \lim_{p\to 1} u(x,t,p) \quad (23)$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{24}$$

where $p \in (0,1]$ is an embedding parameter and the He's polynomials that can be generated by:

$$H_n(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i u_i(x,t))]_{p=0} \ , n = 0,1,2,\dots \tag{25}$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and can be written as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) - T(x,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[f(x,\tau) + L(\sum_{n=0}^{\infty} p^n u_n(x,\tau)) + N(\sum_{n=0}^{\infty} p^n u_n(x,\tau))]d\tau \tag{26}$$

Comparing the terms of same powers of gives solutions of various orders with the first term:

$$u_0(x,t) = T(x,t) \tag{27}$$

We include that the term is the Taylor series of the exact solution of equation (9) of order $n-1$.

5 Applications

In this section we solve some nonlinear partial differential equation with both methods.

Example 5.1: Let's consider the following one dimensional fractional heat- like equation:

$$D_t^{\alpha} u(x,t) = \frac{1}{2}x^2 u_{xx}(x,t) \ , \ 0 < x < 1 \ , \ 0 < \alpha \leq 1 \ , t > 0 \tag{28}$$

With the boundary conditions;

$$u(0,t) = 0 \ , \ \ u(1,t) = e^t$$

and initial condition;

$$u(x,0) = x^2$$

5.1 Application method of Homotopy perturbation Elzaki transform

Apply the steps involved in HPETM as presented in section 4.1 to equation (28) we obtain the following:

$$\mathrm{p}^0 : \mathrm{u}_0(\mathrm{x},\mathrm{t}) = \mathrm{x}^2$$

$$p^1 : u_1(x,t) = E^{-1}\left[v^{\alpha}E\left[\frac{1}{2}x^2 u_0(x,t)_{xx}\right]\right] = E^{-1}\left[v^{\alpha}E[x^2]\right] = E^{-1}[x^2 v^{\alpha+2}] = \frac{x^2 t^{\alpha}}{\alpha!} = \frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)} \ ,$$

$$p^2 : u_2(x,t) = E^{-1}\left[v^{\alpha}E\left[\frac{1}{2}x^2 u_1(x,t)_{xx}\right]\right] = E^{-1}\left[v^{\alpha}E\left[\frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)}\right]\right] = E^{-1}\left[\frac{(v^{2\alpha+2})x^2}{\Gamma(\alpha+1)}\right] = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)} ,$$

Proceeding in a similar way, we have:

$$p^3 : u_3(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_2(x,t)_{xx}\right]\right] = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}x^2 u_n(x,t)_{xx}\right]\right] = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the solution $u(x,t)$ can be written in the form:

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \tag{29}$$

This is an equivalent form to the exact solution in closed form:

$$u(x,t) = x^2 E_\alpha(t^\alpha) \tag{30}$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function

5.2 Application the method of Homotopy perturbation Adomain decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (28) we obtain the following

$$\sum_{n=0}^{\infty} p^n u_n(x,t) - x^2 = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(\sum_{n=0}^{\infty} p^n u_n(x,\tau)_{xx})]d\tau \tag{31}$$

Comparing the terms of the same powers of p we obtain:

$$\mathrm{u_0(x,t) = x^2}$$

$$u_1(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_0(x,\tau)_{xx})]d\tau = \frac{x^2 t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_1(x,\tau)_{xx})]d\tau = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_2(x,\tau)_{xx})]d\tau = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$u_n(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x^2(u_{n-1}(x,\tau)_{xx})]d\tau = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Hence, the asymptotic solution can expressed by

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \tag{32}$$

$$\lim_{\substack{n\to\infty \\ \alpha\to 1}} u_n(x,t,\alpha) = x^2 e^t$$

This is the exact solution of equation (28) when $= 1$.

Example 5.2: Let's Consider the following tow dimensional fractional heat like equation:

$$D_t^\alpha u = u_{xx} + u_{yy},\ \ 0 < x\ ,\ \ y < 2\pi\ ,\ \ 0 < \alpha \le 2\ , t > 0 \tag{33}$$

with the initial conditions

$$u(x,y,0) = \sin x \sin y$$

5.3 Application method of Homotopy perturbation Elzaki transform

Applying the steps involved in HPETM as presented in section 4.1 to equation (33) we obtain the following:

$$p^0 : u_0(x,y,t) = \sin x \sin y$$

$$p^1 : u_1(x,y,t) = E^{-1}\left[v^\alpha E\left[u_{0_{xx}} + u_{0_{yy}}\right]\right] = \frac{-2\sin x \sin y\, t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2 : u_2(x,y,t) = E^{-1}\left[v^\alpha E\left[u_{1_{xx}} + u_{1_{yy}}\right]\right] = \frac{4\sin x \sin y\ \ t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar way, we have:

$$p^3 : u_3(x,y,t) = \frac{-8\sin x \sin y\, t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,t) = \frac{(-2)^n \sin x \sin y\, t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the solution $u(x,t)$ can be written in the form:

$$u(x,y,t) = \sin x \sin y\left(1 - \frac{(2t^\alpha)}{\Gamma(\alpha+1)} + \frac{(2t^\alpha)^2}{\Gamma(2\alpha+1)} - \frac{(2t^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots + \frac{(2t^\alpha)^n}{\Gamma(n\alpha+1)} + \cdots\right) \tag{34}$$

For the special case when $\alpha = 1$, we can get the solution in a closed form

$$u(x,y,t) = e^{-2t}\sin x \sin y \tag{35}$$

5.4 Application the method of Homotopy perturbation Adomain decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (33) we obtain the following

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)_{xx} + \sum_{n=0}^{\infty} p^n u_n(x,y,t)_{yy}\right)\right] d\tau \tag{36}$$

$$p^0 : u_0(x,y,t) = \sin x \sin y$$

$$u_1(x,y,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(u_{0_{xx}} + u_{0_{yy}})d\tau = \frac{(-2\sin x \sin y)t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,y,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(u_{1_{xx}} + u_{1_{yy}})d\tau = \frac{(4\sin x \sin y)t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\vdots$$

$$u_n(x,y,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(u_{n_{xx}} + u_{n_{yy}})d\tau = \frac{(-2)^n(\sin x \sin y)t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the solution $u(x,t)$ can be written in the form:

$$u(x,y,t) = \sin x \sin y\left(1 - \frac{(2t^\alpha)}{\Gamma(\alpha+1)} + \frac{(2t^\alpha)^2}{\Gamma(2\alpha+1)} - \frac{(2t^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots + \frac{(2t^\alpha)^n}{\Gamma(n\alpha+1)} + \cdots\right) \tag{37}$$

For the special case when $\alpha = 1$, we can get the solution in a closed form

$$u(x,y,t) = e^{-2t}\sin x \sin y \tag{38}$$

This is the exact solution for this case

Example 5.3: Let's Consider the following three dimensional fractional heat-like equation:

$$D_t^\alpha u(x,y,z,t) = x^4y^4z^4 + \frac{1}{36}\left(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}\right), 0 < x,y,z < 1, 0 < \alpha \leq 1 \quad (39)$$

With the initial condition;

$$u(x,y,z,t) = 0$$

5.5 Application method of Homotopy perturbation Elzaki transform

Applying the steps involved in HPETM as presented in section 4.1 to equation (39) we obtain the following:

$$p^0 : u_0(x,y,z,t) = x^4y^4z^4$$

$$p^1 : u_1(x,y,z\ t) = E^{-1}\left[v^\alpha E\left[\frac{1}{36}\left(x^2u_{0_{xx}} + y^2u_{0_{yy}} + z^2u_{0_{zz}}\right)\right]\right] = \frac{x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,y,z\ t) = E^{-1}\left[v^\alpha E\left[\frac{1}{36}\left(x^2u_{1_{xx}} + y^2u_{1_{yy}} + z^2u_{1_{zz}}\right)\right]\right] = \frac{x^4y^4z^4t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar way, we have:

$$p^3 : u_3(x,y,z,t) = \frac{x^4y^4z^4t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,z,t) = \frac{x^4y^4z^4t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the solution $u(x,t)$ can be written in the form:

$$u(x,t) = x^4y^4z^4\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (40)$$

5.6 Application the method of Homotopy perturbation Adomain decomposition

Applying the steps involved in HDM as presented in section 4.2 to equation (39) we obtain the following

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}$$

$$\left[x^4y^4z^4\left(\frac{1}{36}\left(x^2\sum_{n=0}^{\infty}p^nu_n(x,y,z,t)_{xx} + y^2\sum_{n=0}^{\infty}p^nu_n(x,y,z,t)_{yy} + z^2\sum_{n=0}^{\infty}p^nu_n(x,y,z,t)_{zz}\right)\right)\right]d\tau \quad (41)$$

$$u_0(x,y,z,t) = 0$$

$$u_1(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(x^4y^4z^4)d\tau = \frac{x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)},$$

$$\vdots$$

$$u_n(x,y,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[(x^4y^4z^4)\frac{1}{36}\left(x^2u_{n-1_{xx}} + y^2u_{n-1_{yy}} + z^2u_{n-1_{zz}}\right)\right]d\tau = \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore, the approximate solution of equation for the first N can be expressed by:

$$u_n(x,y,z,t) = \sum_{n=1}^{N} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)}$$

when $N \to \infty$ the solution can be expressed by

$$u_n(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)} - (x^4y^4z^4) = (x^4y^4z^4)[E_\alpha(t^\alpha) - 1]$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function.

Note that in the case $\alpha = 1$

$$u(x,y,z,t) = (xyz)^4[e^t - 1] \quad (42)$$

This is the exact solution for case of $\alpha = 1$.

6 Conclusion

In this paper, the Homotopy Perturbation Elzaki Transform Method (HPETM) and Homotopy Decomposition Method (HDM) are used to solve nonlinear fractional Heat - Like equations. These two methods are very efficient techniques and quite capable, practically for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HDM has an advantage over the HPETM which is that it solves the nonlinear problems using only the inverse operator which is simple the fractional integral. Also we do not need to use any order inverse transform to find the components of the series solutions as in the case of HPETM. In addition the calculations involved in HDM are very simple and easy execution.

Competing Interests

Authors have declared that no competing interests exist.

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