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The Type I Half Logistic Skew-t Distribution: A Heavy-Tail Model with Inverted Bathtub Shaped Hazard Rate

O. D. Adubisi[1,2*], A. Abdulkadir[2], H. Chiroma[2] and U. F. Abbas[2]

[1]Department of Mathematics and Statistics, Federal University Wukari, Taraba State, Nigeria.
[2]Department of Mathematical Sciences, Abubakar Tafawa Balewa University, Bauchi State, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this article a new generalization of the skew student-t distribution was introduced. The two-parameter model called the type I half-logistic skew-t ($TIHL_{ST}$) distribution can fit skewed, heavy-right tail, and long-tail datasets. Statistical properties of the type I half-logistic skew-t ($TIHL_{ST}$) distribution were derived and the maximum likelihood method parameter estimates assessed through a simulation study. A well-known dataset was analysed, illustrating the usefulness of the new distribution in modeling skewed and heavy-tailed data. The hazard rate shape was found to be increasing, decreasing and inverted bathtub shaped which was also reflected in the application result.

Keywords: Entropy; maximum likelihood estimation; simulation; Skew-t distribution; type I half-logistic distribution.

1 Introduction

The methods of extending the flexibility of various continuous probability distributions are well-known in the literature. Hence, significant efforts in developing new families of flexible continuous probability distributions

**Corresponding author: Email: adubisiobinna@fuwukari.edu.ng;*

have been made by several authors over the years due to the inability of the classical probability models to fit various real-life datasets. Some of the generated families of distributions are: the Gompertz-G family of distributions by Alizadeh et al. [1], Beta-G family of distributions by Eugene et al. [2] and Jones [3], Weibull-G family of distributions by Bourguignon et al. [4], Exponentiated generalized-G family of distributions by Cordeiro et al. [5], Kumaraswamy-G family of distributions by Cordeiro and Castro [6], Gamma-G Type-I family of distributions by Zografos and Balakrishnan [7], Gamma-G Type-II family of distributions by Ristic and Balakrishnan [8], Gamma-X family of distributions by Alzaatreh et al. [9], McDonald-G family of distributions by Alexander et al. [10], Logistic-X family of distributions by Tahir et al. [11], including several others.

The skew-t distribution introduced as an extension of the symmetric t-distribution has been used extensively especially in the field of econometric, time series and financial analysis. Numerous authors have introduced various forms of the skew-t, for example Johnson et al. [12], Azzalini and Capitanio [13], Sahu et al. [14], Gupta [15] and others. Also, several authors have studied possible extensions and generalizations of the skew-t distribution which include the Kumaraswamy skew-t distribution by Khamis et al. [16], Balakrishnan skew-t distribution by Shafiei and Doostparast [17], generalized hyperbolic skew-t distribution by Aas and Haff [18], Beta skew-t distribution by Shittu et al. [19], Exponentiated skew-t by Dikko and Agboola [20] and beta skew-t distribution by Basalamah et al. [21].

This article focuses on extending the skew-t distribution by adding a parameter (shape) to increase its flexibility and efficacy to real-life data sets. The motivation in developing the new distribution is to create a flexible heavy-tailed distribution with right-skewed, and unimodal features. The proposed distribution can serve as an alternative error innovation in modeling and forecasting financial return series using GARCH models. This article is structured as follows: In section 2, the new distribution called the $\mathrm{TIHL}_{\mathrm{ST}}$ distribution is introduced. Section 3; presents some statistical properties of the $\mathrm{TIHL}_{\mathrm{ST}}$ distribution. In Section 4, we have estimates of the unknown parameters using the maximum likelihood estimation procedure and the simulation study. In section 5, we illustrate the usefulness of the $\mathrm{TIHL}_{\mathrm{ST}}$ distribution using two real-life datasets. Conclusion in section 6.

2 Type I Half-Logistic Skew-T Distribution

Jones [22], and Jones and Faddy [23] established a tractable skewed extension of the symmetric student-t distribution known as the skew student-t (skew-t) distribution. The skew-t distribution cumulative distribution function (CDF) is given as

$$G_{ST}(y)=\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right),\quad \eta>0, y\in(-\infty,\infty) \tag{1}$$

The probability distribution function (PDF) obtained by differentiating (1) is given as

$$g_{ST}(y)=\frac{\lambda}{2\left(\eta+y^2\right)^{3/2}} \tag{2}$$

where η is the skew parameter.

Cordeiro et al. [24] introduced the CDF of type-I half-logistic family of distributions which is expressed as

$$F(y,\varphi,\kappa)=\int_0^{-\log\left[1-G(y,\eta)\right]}\frac{2\varphi e^{-\varphi y}}{\left(1+e^{-\varphi y}\right)^2}dy=\frac{\left(1-\left[1-G(y;\kappa)\right]^{\varphi}\right)}{\left(1+\left[1-G(y;\kappa)\right]^{\varphi}\right)}, \tag{3}$$

The PDF by differentiating (3) is given as:

$$f(y,\varphi,\kappa)=\frac{2\varphi g(y;\kappa)\left\langle\left[1-G(y;\kappa)\right]^{\varphi-1}\right\rangle}{\left\{1+\left[1-G(y;\kappa)\right]^{\varphi}\right\}^{2}}, \tag{4}$$

where $\varphi>0$ is the shape parameter, $G(y;\kappa)$ and $g(y;\kappa)$ are the parent distribution CDF and PDF depending on the parameter (κ) vector. A two-parameter model called the type I half-logistic skew-t (TIHL$_{ST}$) distribution is proposed. The PDF of the TIHL$_{ST}$ distribution is obtained by inserting Equations (1) and (2) into Equation (4):

$$f(y,\upsilon)=\frac{2\varphi\left(\frac{\eta}{2\left(\eta+y^{2}\right)^{3/2}}\right)\left\langle\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi-1}\right\rangle}{\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}\right\}^{2}},\quad \varphi,\eta>0,\ y\in(-\infty,\infty) \tag{5}$$

The corresponding CDF by inserting Equation (1) into Equation (3) is given as:

$$F(y,\upsilon)=\left\langle\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}\right\rangle,\varphi,\eta>0,\ y\in(-\infty,\infty) \tag{6}$$

where φ is the shape parameter and η is the skew parameter.

The survival function is defined as $s(y)=1-F(y)$, given a random variable Y. Hence, the survival function $s(y)$ of TIHL$_{ST}$ distribution is given as:

$$s(y)=\frac{2\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}}{\left(1+\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}\right)}$$

The hazard rate function $h(y)$, reversed hazard rate function $r(y)$, cumulative hazard rate function $H(y)$ and odds function $O(y)$ are respectively, given as:

$$h(y)=\frac{\varphi\eta}{2\left(\eta+y^{2}\right)^{3/2}\left\{1+\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}\right\}\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)},$$

$$r(y)=\frac{\varphi\eta\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi-1}}{\left(\eta+y^{2}\right)^{3/2}\left\{1-\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{2\varphi}\right\}},$$

$$\mathrm{H}(y)=-\ln\left[\frac{2\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}}{\left(1+\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}\right)}\right],$$

$$O(y)=\frac{1-\left(\frac{1}{2}-\frac{y}{2\sqrt{\eta+y^{2}}}\right)^{\varphi}}{\left(1-\frac{y}{\sqrt{\eta+y^{2}}}\right)^{\varphi}}$$

To show the efficacy of the TIHL_{ST} distribution, Fig. 1 and Fig. 2 presents the PDF plot and hazard rate plot for some designated values of the parameters. We observed from the graphs in Fig. 1 that the PDF is symmetrical, right-skewed and heavy-tail depending on the chosen parameter values while the hazard rate function is increasing, decreasing, and inverted bathtub shaped as depicted in Fig. 2.

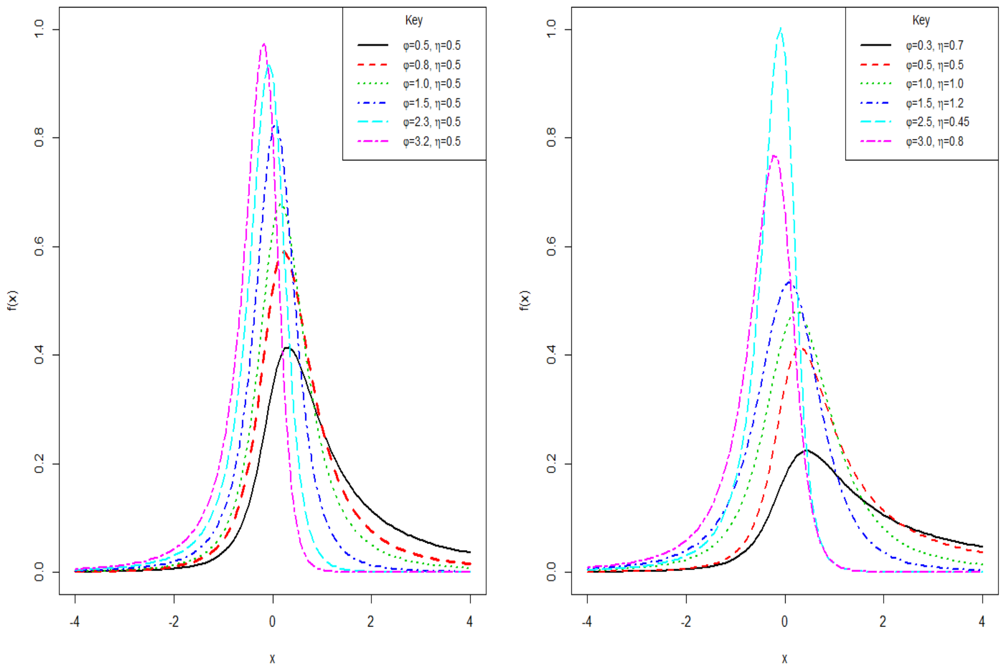

Fig. 1. PDF plots of the TIHL_{ST} for designated values of the parameters

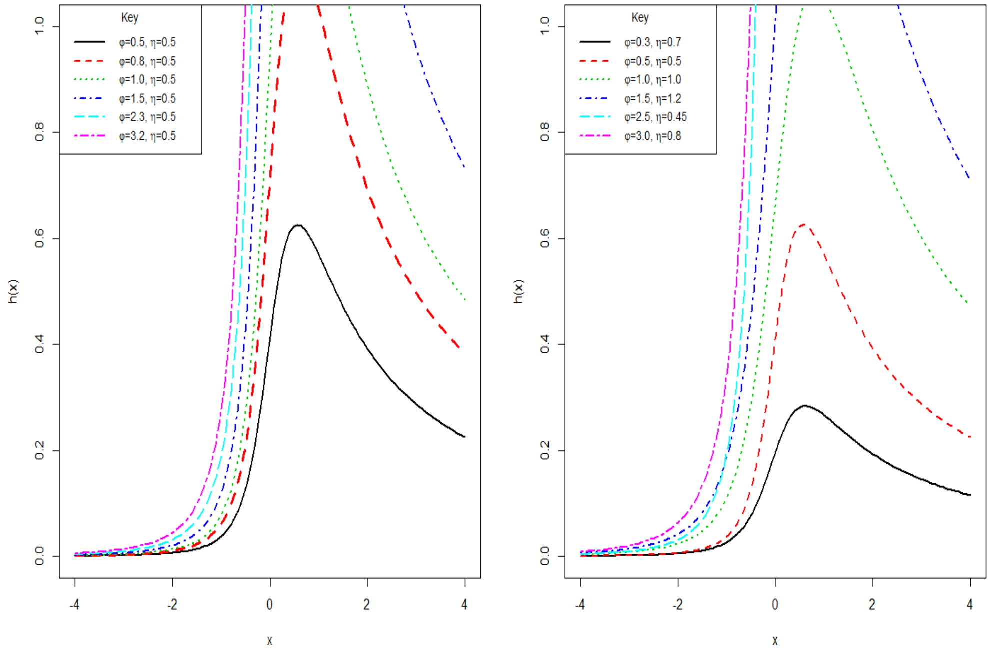

Fig. 2. Hazard rate function plots of the TIHL$_{ST}$ for designated values of the parameters

3 Statistical Properties

In this section, we derive the structural properties of the TIHL$_{ST}$ distribution.

3.1 Quantile function

The quantile function $Q(u) = F(y)^{-1}$ for $u \in (0,1)$ of the TIHL$_{ST}$ distribution is given by:

$$Q(u) = \eta^{\frac{1}{2}} \frac{\left[1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right]}{\left\{\left[1-\left(1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right)^{2}\right]^{\frac{1}{2}}\right\}}, \qquad u \in (0,1). \tag{7}$$

The median $Q(0.5)$ is derived by setting $u = 0.5$ in (7). Moreso, the other quantiles can be derived similarly by setting $u = 0.25$ and $u = 0.75$.

$$Q(0.5) = \eta^{\frac{1}{2}} \frac{\left[1-2\left(\frac{1-(0.5)}{1+(0.5)}\right)^{\frac{1}{\varphi}}\right]}{\left\{\left[1-\left(1-2\left(\frac{1-(0.5)}{1+(0.5)}\right)^{\frac{1}{\varphi}}\right)^{2}\right]^{\frac{1}{2}}\right\}}, \qquad u \in (0,1). \tag{8}$$

We can use the $TIHL_{ST}$ quantile function (7) for generating random values from the $TIHL_{ST}$ distribution. The Bowley skewness by Kenney and Keeping [25], and Moor's kurtosis by Moor [26] are as follows:

$$S_k = \frac{Q\left(\frac{3}{4};\varphi,\eta\right) - 2Q\left(\frac{1}{2};\varphi,\eta\right) + Q\left(\frac{1}{4};\varphi,\eta\right)}{Q\left(\frac{3}{4};\varphi,\eta\right) - Q\left(\frac{1}{4};\varphi,\eta\right)}$$

$$K = \frac{Q\left(\frac{7}{8};\varphi,\eta\right) - Q\left(\frac{5}{8};\varphi,\eta\right) - Q\left(\frac{3}{8};\varphi,\eta\right) + Q\left(\frac{1}{8};\varphi,\eta\right)}{Q\left(\frac{6}{8};\varphi,\eta\right) - Q\left(\frac{2}{8};\varphi,\eta\right)}$$

where $Q(.)$ represent the quantile function. Using the $TIHL_{ST}$ quantile function (7), the numeric values of the median (M), 25th and 75th percentiles, interquartile range (IQR), kurtosis (Ks), and skewness (Sk) for some chosen parameter values are provided in Table 1. It is clear that as the values of η increases at specific values of φ; the median, 25th, and 75th percentiles, and IQR increases while the skewness and kurtosis remain constant. Moreso, across different values of φ, the skewness and kurtosis decreases indicating positive and negative properties, respectively.

Table 1. Descriptive statistics of the $TIHL_{ST}$ distribution

φ	η	**M**	**25th**	**75th**	**Sk**	**Ks**	**IQR**
0.2	0.3	4.243	0.864	35.501	0.805	6.422	34.637
	0.5	5.477	1.115	45.831	0.805	6.422	44.716
	0.9	7.348	1.496	61.689	0.805	6.422	59.993
	1.2	8.485	1.727	71.001	0.805	6.422	69.274
	1.5	9.487	1.931	79.382	0.805	6.422	77.451
	2.0	10.954	2.230	91.662	0.805	6.422	89.433
0.4	0.3	0.974	0.270	3.082	0.499	1.865	2.811
	0.5	1.258	0.349	3.979	0.499	1.865	3.630
	0.9	1.687	0.468	5.338	0.499	1.865	4.870
	1.2	1.949	0.540	6.164	0.499	1.865	5.624
	1.5	2.179	0.604	6.892	0.499	1.865	6.288
	2.0	2.516	0.697	7.958	0.499	1.865	7.260
0.6	0.3	0.507	0.081	1.303	0.303	0.876	1.222
	0.5	0.655	0.105	1.683	0.303	0.876	1.578
	0.9	0.879	0.140	2.258	0.303	0.876	2.117
	1.2	1.015	0.162	2.607	0.303	0.876	2.445
	1.5	1.134	0.181	2.915	0.303	0.876	2.734
	2.0	1.310	0.209	3.366	0.303	0.876	3.156
0.7	0.3	0.394	0.020	0.994	0.232	0.621	0.975
	0.5	0.508	0.025	1.284	0.232	0.621	1.258
	0.9	0.682	0.034	1.722	0.232	0.621	1.688
	1.2	0.787	0.040	1.989	0.232	0.621	1.949
	1.5	0.880	0.044	2.223	0.232	0.621	2.179
	2.0	1.017	0.051	2.567	0.232	0.621	2.516
1.5	0.3	0.021	-0.255	0.279	-0.036	-0.083	0.534
	0.5	0.027	-0.330	0.360	-0.036	-0.083	0.690
	0.9	0.036	-0.442	0.483	-0.036	-0.083	0.925
	1.2	0.042	-0.511	0.557	-0.036	-0.083	1.068
	1.5	0.047	-0.571	0.623	-0.036	-0.083	1.194
	2.0	0.054	-0.660	0.719	-0.036	-0.083	1.379

3.2 Asymptotic behaviour

The limits of the TIHL$_{ST}$ density function (PDF) are given by

$$\lim_{y\to-\infty} f(y) = \lim_{y\to+\infty} f(y) = 0$$

Proof:
For $y \to \infty$, we have

$$\lim_{y\to\infty} f(x) = \lim_{y\to\infty}\left(2\varphi\left(\frac{\eta}{2\left(\eta+y^2\right)^{3/2}}\right)\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\right]^{\varphi-1}\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\right]^{\varphi}\right\}^{-2}\right), \quad = 0 \tag{9}$$

Similarly, for $y \to -\infty$, we have

$$\lim_{y\to-\infty} f(x) = \lim_{t\to-\infty}\left(2\varphi\left(\frac{\eta}{2\left(\eta+y^2\right)^{3/2}}\right)\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\right]^{\varphi-1}\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\right]^{\varphi}\right\}^{-2}\right), \quad = 0 \tag{10}$$

The results of the asymptotic behaviour infer the THIL$_{ST}$ mode is unique and presented fully in the Appendix.

3.3 Mixture representations

The series expansion of the TIHL$_{ST}$ distribution is derived for the density and cumulative functions. If $|s|<1$ and k a positive real non-integer, the generalized binomial theorem representation is given by:

$$(1-s)^{k-1} = \sum_{j=0}^{\infty}(-1)^j\binom{k-1}{j}s^j \tag{11}$$

According to Cordeiro et al. [24], expansion of the PDF, applying the series expansion (11) in (5) leads to

$$f(y,\upsilon) = \sum_{c=0}^{+\infty}\sum_{d=0}^{+\infty} b_{c,d}\mathrm{P}_d(y;\eta) \tag{12}$$

where, $b_{c,d} = (-1)^{c+d}\, 2\varphi(c+1)\binom{\varphi(c+1)-1}{d}$ and $\mathrm{P}_d(y,\eta) = \left(\frac{\eta}{2\left(\eta+y^2\right)^{3/2}}\right)\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^d$.

$f(y,\upsilon)$ reveals the PDF expression is likely an infinite linear combination of the skew-t density-functions. Thus, we can obtain the statistical properties of the TIHL$_{ST}$ distribution from the properties of the skew-t distribution. Also, another expanded form of the PDF is given by

$$f(y,\upsilon) = w_{c,d,e}\, y^e\left(\eta+y^2\right)^{-\left(\frac{e+3}{2}\right)} \tag{13}$$

where, $w_{c,d,e} = \frac{\varphi\eta}{2^d}\sum_{c,d=0}^{\infty}\sum_{e=0}^{d}(-1)^{c+d}(c+1)\binom{\varphi(c+1)-1}{d}\binom{d}{e}$.

The CDF of the TIHL$_{ST}$ distribution by simplifying (6), is given by

$$F(y,\upsilon) = -1 + 2\frac{1}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\right]^{\varphi}} \tag{14}$$

Using (14), the expansion of $\left[F(y,\upsilon)\right]^s$, where s is positive integer, is given by

$$\left[F(y,\upsilon)\right]^s = \sum_{q=0}^{s}\sum_{w,z=0}^{+\infty} \vartheta_{q,w,z} G_z\left(y;\eta\right) \tag{15}$$

where, $\vartheta_{q,w,z} = \left(-1\right)^{s-q+z} 2^q \binom{s}{q}\binom{-q}{w}\binom{\varphi w}{z}$ and $G_z(y,\eta) = \left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^z$ denote the cumulative density function of the skew-t distribution with power $z > 0$. Another expanded form of $\left[F(y,\upsilon)\right]^s$ is given by

$$\left[F(y,\upsilon)\right]^s = \vartheta_{q,w,z,u} y^u \left(\eta + y^2\right)^{-\left(\frac{u}{2}\right)} \tag{16}$$

where, $\vartheta_{q,w,z,u} = \sum_{q=0}^{s}\sum_{w,z=0}^{+\infty}\sum_{u=0}^{z}(-1)^{s-q+z} 2^{q-z} \binom{s}{q}\binom{-q}{w}\binom{\varphi w}{z}\binom{z}{u}$

3.4 Moments

Let Y be a random variable which follows the $TIHL_{ST}\left(\varphi,\eta\right)$, then the gth raw moment of Y is given by

$$\mu_g' = \int_{-\infty}^{+\infty} y^g w_{c,d,e} y^e \left(\eta + y^2\right)^{-\left(e+3/2\right)} dy \tag{17}$$

Taboga [27] showed that (17) can be rewritten as:

$$\mu_g' = \left(1+\left(-1\right)^g\right) w_{c,d,e} \int_0^{+\infty} y^{g+e} \left(\eta + y^2\right)^{-\left(e+3/2\right)} dy \tag{18}$$

After some algebra, the g^{th} moment of Y, using the Beta function expression $B\left(\theta,\gamma\right) = \int_0^{+\infty} y^{\theta-1}\left(1+y\right)^{-\theta-\gamma} dy$ is given by

$$\mu_g' = \begin{cases} w_{c,d,e}\eta^{\frac{g-2}{2}} \mathrm{B}\left(\frac{g+e+1}{2},\frac{2-g}{2}\right) & g = even \\ 0 & g = odd \end{cases} \tag{19}$$

where $w_{c,d,e} = \frac{\varphi\eta}{2^d}\sum_{c,d=0}^{\infty}\sum_{e=0}^{d}(-1)^{c+d}\left(c+1\right)\binom{\varphi\left(c+1\right)-1}{d}\binom{d}{e}$

Let Y be a random variable which follows the $TIHL_{ST}\left(\varphi,\eta\right)$, then the g^{th} incomplete moment for any $t > 0$ is given by

$$\varphi_g'\left(t\right) = \int_0^t y^g w_{c,d,e} y^e \left(\eta + y^2\right)^{-\left(e+3/2\right)} dy \tag{20}$$

After some algebra, the r^th^ incomplete moment of Y, using the Beta function expression $B(z,\theta,\gamma)=\int_0^z y^{\theta-1}(1-y)^{\gamma-1}\,dy$ is given by

$$\varphi_g'(t)=w_{c,d,e}^{*}\eta^{\frac{g-2}{2}}\mathrm{B}\left(t,\frac{g+e+1}{2},\frac{2-g}{2}\right) \tag{21}$$

where $w_{c,d,e}^{*}=\frac{\varphi\eta}{2^d}\sum_{c,d=0}^{\infty}\sum_{e=0}^{d}(-1)^{c+d}(c+1)\binom{\varphi(c+1)-1}{d}\binom{d}{e}$

Remark: The first incomplete moment $\varphi_1'(t)=\int_0^t yf(y)\,dy$ of TIHL$_{ST}$ distribution can be obtained by inserting $g=1$ in (21).

3.5 Probability weighted moments

An important mathematical quantity is the probability weighted moment (PWM). The PWM $\tau_{g,s}$ of a random variable Y is given by

$$\tau_{g,s}=\mathrm{E}\left[Y^g F(y)^s\right]=\int_{-\infty}^{+\infty} y^g f(y)\left(F(y)\right)^s dy \tag{22}$$

Inserting (13) and (16) in (22) using the expression by Taboga [27], the PWM of the TIHL$_{ST}$ is given as:

$$\tau_{g,s}=\left(1+(-1)^g\right)\int_0^{+\infty} A^{*}x^{g+e+u}\left(\eta+y^2\right)^{-\left(\frac{e+u+3}{2}\right)} dy \tag{23}$$

where $A^{*}=w_{c,d,e}\vartheta_{q,w,z,u}$

After some algebra, the PWM of the TIHL$_{ST}$, using the Beta function expression $B(\theta,\gamma)=\int_0^{+\infty} y^{\theta-1}(1+y)^{-\theta-\gamma}\,dy$ is given by

$$\tau_{g,s}=\begin{cases} A^{*}\eta^{\frac{g-2}{2}}\mathrm{B}\left(\frac{g+e+u+1}{2},\frac{2-g}{2}\right) & g=even \\ 0 & g=odd \end{cases} \tag{24}$$

3.6 Order statistics

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample from a continuous distribution and $Y_{1:n}<Y_{2:n}<\ldots<Y_{n:n}$ are the order statistics obtained from the sample. The r^th^ order statistic $Y_{r:n}$ is defined as

$$f_{r:n}(y)=\frac{g(y)}{\mathrm{B}(r,n-r+1)}\left[G(y)\right]^{r-1}\left[1-G(y)\right]^{n-r} \tag{25}$$

where $r>0$, $y\in(-\infty,\infty)$, $G(y)$ and $g(y)$ are the CDF and PDF of TIIHL$_{ST}$ distribution, $\mathrm{B}(.,.)$ represent the beta function expression. Given that $0<G(y)<1$ for $y>0$, the expression in (25) can be rewritten as:

$$f_{r:n}(y)=\frac{1}{\mathrm{B}(r,n-r+1)}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\left[G(y)\right]^{r+l-1}g(y) \tag{26}$$

Inserting (5) and (6) in (26), applying series expansion. The r[th] order statistics for $TIHL_{ST}$ distribution is given as

$$f_{r:n}(y)=\frac{1}{\mathrm{B}(r,n-r+1)}\vartheta_{l,c,d,v,z}y^{z}\left(\eta+y^{2}\right)^{-\left(\frac{z+3}{2}\right)} \tag{27}$$

where, $\vartheta_{l,c,d,v,z}=\frac{\varphi\eta}{2^{v}}\sum_{l=0}^{n-r}\sum_{c=o}^{r+l-1}\sum_{d,v=0}^{\infty}\sum_{z=0}^{v}(-1)^{l+c+d+v}\binom{n-r}{l}\binom{r+l-1}{c}\binom{r+l+d}{d}\binom{\varphi(c+d+1)-1}{v}\binom{v}{z}$

Remark: The smallest and largest order statistics is derived by setting $r=1$ and $r=n$ in (27). Therefore, the smallest order statistics is expressed as

$$f_{1:n}(y)=\frac{1}{\mathrm{B}(1,n)}\vartheta_{l,c,d,v,z}y^{z}\left(\eta+y^{2}\right)^{-\left(\frac{z+3}{2}\right)} \tag{28}$$

where, $\vartheta_{l,c,d,v,z}=\frac{\varphi\eta}{2^{v}}\sum_{l=0}^{n-1}\sum_{c=o}^{l}\sum_{d,v=0}^{\infty}\sum_{z=0}^{v}(-1)^{l+c+d+v}\binom{n-1}{l}\binom{l}{c}\binom{1+l+d}{d}\binom{\varphi(c+d+1)-1}{v}\binom{v}{z}$

The largest order statistics is expressed as

$$f_{n:n}(y)=\frac{1}{\mathrm{B}(n,1)}\vartheta_{l,c,d,v,z}y^{z}\left(\eta+y^{2}\right)^{-\left(\frac{z+3}{2}\right)} \tag{29}$$

where, $\vartheta_{l,c,d,v,z}=\frac{\varphi\eta}{2^{v}}\sum_{l=0}^{n-n}\sum_{c=o}^{n+l-1}\sum_{d,v=0}^{\infty}\sum_{z=0}^{v}(-1)^{l+c+d+v}\binom{n-n}{l}\binom{n+l-1}{c}\binom{n+l+d}{d}\binom{\varphi(c+d+1)-1}{v}\binom{v}{z}$

3.7 Entropies

The variation of uncertainty in a random variable is normally measured by the entropy. The Rényi entropy $\mathrm{I}_{R(\delta)}$ is expressed as:

$$\mathrm{I}_{R(\delta)}=\frac{1}{1-\delta}\log\int_{-\infty}^{+\infty}f(y)^{\delta}\,dy,\qquad \delta>0 \text{ and } \delta\neq 1 \tag{30}$$

Using the PDF mixture representation of $TIHL_{ST}$ distribution in (13), $f(y)^{\delta}$ is given as:

$$f(y)^{\delta}=w_{c,d,e}x^{e}\left(\eta+y^{2}\right)^{-\left(e+3\delta/2\right)} \tag{31}$$

where $w_{c,d,e}=\frac{(\varphi\eta)^{\delta}}{2^{d}}\sum_{c,d=0}^{\infty}\sum_{e=0}^{d}(-1)^{c+d}\binom{2\delta+c-1}{c}\binom{\varphi(c+\delta)-\delta}{d}\binom{d}{e}$

Hence, the Rényi entropy of the $TIHL_{ST}$ distribution using the expression by Taboga [27], is expressed as:

$$\mathrm{I}_{R(\delta)}=\frac{1}{1-\delta}\log\left(\left(1+(-1)^{g}\right)w_{c,d,e}\int_{0}^{+\infty}y^{e}\left(\eta+y^{2}\right)^{-\left(e+3\delta/2\right)}dy\right) \tag{32}$$

Using the expression of the Beta function $B(\theta,\gamma)=\int_0^{+\infty} y^{\theta-1}(1+y)^{-\theta-\gamma}\,dy$. The Rényi entropy of the TIHL$_{ST}$ distribution is given as:

$$\mathrm{I}_{R(\delta)}=\frac{1}{1-\delta}\log\begin{cases} w_{c,d,e}\eta^{\frac{1-3\delta}{2}}\mathrm{B}\left(\frac{e+1}{2},\frac{3\delta-1}{2}\right) & g=even \\ 0 & g=odd\end{cases} \tag{33}$$

Furthermore, the q-entropy is defined as

$$\mathrm{H}_{\delta}=\frac{1}{1-\delta}\log\left(1-\int_{-\infty}^{+\infty} f(y)^{\delta}\,dy\right),\ \delta>0 \text{ and } \delta\neq 0 \tag{34}$$

where $\delta=q$

Hence, the q-entropy of TIHL$_{ST}$ distribution is given as

$$\mathrm{H}_{\delta}=\frac{1}{\delta-1}\log\left\{1-\left(\begin{cases} w_{c,d,e}\eta^{\frac{1-3\delta}{2}}\mathrm{B}\left(\frac{e+1}{2},\frac{3\delta-1}{2}\right) & g=even \\ 0 & g=odd\end{cases}\right)\right\} \tag{35}$$

4 Model Estimation

4.1 Parameters estimation

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample from the TIHL$_{ST}$ distribution with unknown parameter vector $\upsilon=(\varphi,\eta)^T$. The log-likelihood function, say l, is given as:

$$\begin{aligned} l=\log L(\upsilon)&=n\log 2\varphi+n\log\eta-n\log 2-3/2\sum_{i=1}^{n}\log\left(\eta+y^2\right)+(\varphi-1)\sum_{i=1}^{n}\log\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right) \\ &-2\sum_{i=1}^{n}\log\left(1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}\right)\end{aligned} \tag{36}$$

Taking the partial derivative of the log-likelihood l, with respect to φ and η equating to zero, the following normal equations are obtained as follows:

$$\frac{\partial l}{\partial\varphi}=\frac{n}{\varphi}+\sum_{i=1}^{n}\ln\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)-2\sum_{i=1}^{n}\frac{\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}\ln\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)}{\left\{1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\varphi+y^2}}\right)\right)^{\alpha}\right\}}=0 \tag{37}$$

$$\frac{\partial l}{\partial \eta}=\frac{n}{\eta}-\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\eta+y^2\right)}+\left(\varphi-1\right)\sum_{i=1}^{n}\frac{y}{4\left(\eta+y^2\right)^{3/2}\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)}$$
$$-\varphi\sum_{i=1}^{n}\frac{x\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}}{2\left(\eta+y^2\right)^{3/2}\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\left\{1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\alpha}\right\}}=0 \tag{38}$$

The non-linear equations (37) and (38) are solved numerically via iterative methods using statistical software such as R, MATLAB, Maple. The maximum likelihood estimates (MLEs) are asymptotic normally distributed i.e., $\sqrt{n}\left(\hat{\alpha}-\alpha,\hat{\lambda}-\lambda\right) \sim \mathrm{N}_2\left(0,\Sigma\right)$, where Σ is the variance-covariance matrix obtained by inverting the observed Fisher information $\left(F\right)$ given as follows:

$$F=\begin{bmatrix}\frac{\partial^2 l}{\partial \varphi^2} & \frac{\partial^2 l}{\partial \varphi \partial \eta}\\ \frac{\partial^2 l}{\partial \varphi \partial \eta} & \frac{\partial^2 l}{\partial \eta^2}\end{bmatrix}$$

For each parameter of TIHL_{ST} distribution, the asymptotic $\left(1-\tau\right)100\%$ confidence intervals are estimated with

$$\hat{\varphi}\pm Z_{\tau/2}\sqrt{\Sigma_{11}}$$

$$\hat{\eta}\pm Z_{\tau/2}\sqrt{\Sigma_{22}}$$

where, upper τ^{th} percentile of the standard normal distribution is Z_{τ}.

4.2 Simulations study

In this section, the efficiency and flexibility of the TIHL_{ST} distribution is appraised using simulation study. The simulation is carried out as follows:

- Data are generated using the quantile function of the TIHL_{ST} distribution.

$$Y=\eta^{\frac{1}{2}}\frac{\left[1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right]}{\left[1-\left(1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right)^2\right]^{\frac{1}{2}}}$$

where $\left(u\right)$ is uniform random numbers with parameter $\left(0,1\right)$.

- The selected parameter values are set as follows: $\left(\varphi,\eta\right)=\left(1.2,0.7\right),\left(1.5,1.0\right),\left(1.7,1.2\right),\left(2.0,1.5\right)$
- The selected sample sizes are $n=30,50,150,250,300$ and 1000.
- Generated 10,000 samples for each sample size.

The performance of the estimates is evaluated through the average estimates (AE), absolute bias, variance, mean square errors (MSE) and root mean square errors (RMSE) for the different sample sizes. The absolute bias, MSE and RMSE are computed for $\hat{S}=\left(\hat{\varphi},\hat{\eta}\right)$ using

$$AbsBias_s = \left|\frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)\right|$$

$$MSE_s = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2$$

$$RMSE_s = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2}$$

The simulation results for the average MLEs, absolute bias, variance, MSEs, and RMSEs for different combinations of the parameters φ and η are given in Table 2. These estimates are sensibly consistent and approach the parameter values as the sample size increases. The absolute bias, variance, MSEs and RMSEs decrease for all parameter mixtures as the sample size increases which implies that the TIHL$_{ST}$ parameter estimates are very much closer and the maximum likelihood method better estimated the true parameter values as the sample size increases.

Table 2. Mean, absolute bias, variance, RMSE and MSE

			$\left(\varphi=1.2,\eta=0.7\right)$			
n	**Par**	**Mean**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	φ	1.2194	0.0194	0.0563	0.0566	0.2380
	η	0.7384	00384	0.1354	0.1369	0.3700
50	φ	1.2101	0.0101	0.0326	0.0327	0.1809
	η	0.7210	0.0210	0.0777	0.0781	0.2795
150	φ	1.2022	0.0022	0.0107	0.0107	0.1033
	η	0.7060	0.0060	0.0240	0.0240	0.1549
250	φ	1.2018	0.0018	0.0063	0.0063	0.0796
	η	0.7045	0.0045	0.0140	0.0140	0.1183
300	φ	1.2016	0.0016	0.0053	0.0053	0.0727
	η	0.7036	0.0036	0.0117	0.0118	0.1084
1000	φ	1.2006	0.0006	0.0016	0.0016	0.0399
	η	0.7017	0.0017	0.0035	0.0035	0.0591

			$\left(\varphi=1.5,\eta=1.0\right)$			
n	**Par**	**Mean**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	φ	1.5235	0.0235	0.0781	0.0786	0.2804
	η	1.0349	0.0349	0.2253	0.2265	0.4760
50	φ	1.5122	0.0122	0.0450	0.0450	0.2125
	η	1.0185	0.0185	0.1313	0.1317	0.3629
150	φ	1.5025	0.0025	0.0147	0.0147	0.1211
	η	1.0046	0.0046	0.0408	0.0408	0.2021
250	φ	1.5019	0.0019	0.0086	0.0086	0.0929
	η	1.0036	0.0036	0.0238	0.0239	0.1544
300	φ	1.5018	0.0018	0.0072	0.0072	0.0848
	η	1.0029	0.0029	0.0199	0.0199	0.1412
1000	φ	1.5006	0.0006	0.0022	0.0022	0.0465
	η	1.0016	0.0016	0.0059	0.0059	0.0769

$(\varphi = 1.7, \eta = 1.2)$						
n	**Par**	**Mean**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	φ	1.7287	0.0287	0.0939	0.0947	0.3078
	η	1.2321	0.0321	0.2924	0.2935	0.5417
50	φ	1.7150	0.0150	0.0539	0.0541	0.2326
	η	1.2165	0.0165	0.1716	0.1719	0.4146
150	φ	1.7033	0.0033	0.0175	0.0175	0.1322
	η	1.2036	0.0036	0.0536	0.0536	0.2315
250	φ	1.7023	0.0023	0.0102	0.0102	0.1012
	η	1.2030	0.0030	0.0313	0.0313	0.1770
300	φ	1.7022	0.0022	0.0085	0.0085	0.0923
	η	1.2025	0.0025	0.0261	0.0261	0.1617
1000	φ	1.7007	0.0007	0.0026	0.0026	0.0506
	η	1.2014	0.0014	0.0077	0.0077	0.0880

$(\varphi = 2.0, \eta = 1.5)$						
n	**Par**	**Mean**	**AbsBias**	**Var**	**MSE**	**RMSE**
30	φ	2.0397	0.0397	0.1212	0.1227	0.3504
	η	1.5275	0.0275	0.4073	0.4081	0.6388
50	φ	2.0213	0.0213	0.0687	0.0692	0.2630
	η	1.5134	0.0134	0.2402	0.2404	0.4903
150	φ	2.0051	0.0051	0.0221	0.0221	0.1486
	η	1.5024	0.0024	0.0754	0.0754	0.2746
250	φ	2.0033	0.0033	0.0129	0.0129	0.1135
	η	1.5021	0.0021	0.0442	0.0442	0.2102
300	φ	2.0031	0.0031	0.0107	0.0107	0.1035
	η	1.5018	0.0018	0.0367	0.0367	0.1917
1000	φ	2.0008	0.0008	0.0032	0.0032	0.0567
	η	1.5012	0.0012	0.0109	0.0109	0.1043

5 Applications

To illustrate the flexibility and efficacy of the $TIHL_{ST}$ distribution. The dataset on ordered failure of components: 0.0418, 0.0473, 0.0834, 0.1091, 0.2031, 0.2099, 0.004, 0.6143, 0.2918, 0.3465, 0.4035, 0.0142, 0.0221, 0.0009, 0.2168, 0.0261, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.6143, previously used by Ramadan et al. [28] is analysed. The descriptive statistics of the dataset are provided in Table 3. It is obvious that the first and second datasets are highly positively skewed.

Table 3. Descriptive statistics of the first and second datasets

	n	**Mean**	**Median**	**Standard deviation**	**Skewness**	**Kurtosis**
First data	20	0.161	0.133	0.157	1.330	1.514

The $TIHL_{ST}$ distribution is compared with other competitive distributions such as the half logistic skew-t (HLST), skew-t (ST), and Fréchet (FT) distributions. The performance measures are applied using the R-software package "AdequacyModel" to evaluate the fit of the distributions specified above. The distribution parameters are estimated using the maximum likelihood estimation procedure. The following performance measures: Hannan-Quinn information criterion (HQIC), log-likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) including Anderson Darling (AD), Cramer-von Mises (CVM), Kolmogorov-Smirnov (K-S) statistic and their p-values are provided in Tables 4 and 5. The distribution is of a good fit if all the performance measures are smaller and the p-values are larger. Lastly, Table 6 presents the $TIHL_{ST}$ model parameter 95% and 99% confidence intervals for the dataset.

Table 4. MLEs (SE) and performance measures for the dataset

Model	MLE	AIC	CAIC	BIC	HQIC	Rank
$TIHL_{ST}$	$\hat{\varphi} = 0.4694$ (0.1191) $\hat{\eta} = 0.0029$ (0.0022)	-15.886	-15.179	-13.894	-15.497	1
HL_{ST}	$\hat{\eta} = 0.0248$ (0.0119)	-12.579	-12.356	-11.583	-12.384	3
ST	$\hat{\eta} = 0.0393$ (0.0194)	-1.588	-1.365	-0.592	-1.393	4
FT	$\hat{\varphi} = 0.5160$ (0.0781) $\hat{\eta} = 0.0321$ (0.0148)	-13.858	-13.152	-11.866	-13.469	2

Table 5. Performance measures for the dataset

Model	LL	CVM	p-value (CVM)	AD	p-value (AD)	KS	p-value (KS)
$TIHL_{ST}$	9.943	0.208	0.7	1.39	0.6	0.202	0.34
HL_{ST}	7.289	0.362	0.31	2.06	0.31	0.336	0.016
ST	1.794	0.706	0.035	2.96	0.12	0.502	3.412e-05
FT	8.929	0.216	0.69	1.67	0.46	0.196	0.300

From the results in Tables 4 and 5, the performance measures of the $TIHL_{ST}$ distribution are smaller when compared to other fitted distributions, so we infer that the $TIHL_{ST}$ distribution provides a better fit than the other distributions. The flexibility and fitness of the $TIHL_{ST}$ distribution is visible from Fig. 4. It is clear that $TIHL_{ST}$ distribution provides an appropriate fit for the dataset based on the density function, distribution function and P-P plot in Fig. 4.

Furthermore, the hazard rate plot of the $TIHL_{ST}$ distribution, using the parameter estimates in Table 4 is also depicted in Figure 4. The hazard rate shape based on the OE_{ST} parameter estimates is increasing, decreasing and inverted bathtub shaped. The results in Table 6, shows that the parameter estimates fall within the 95% and 99% confidence intervals.

Table 6. $TIHL_{ST}$ distribution parameters confidence intervals for the dataset

CI	φ	η
95%	$[0.2353 \quad 0.7069]$	$[-0.0015 \quad 0.0075]$
99%	$[0.1619 \quad 0.7803]$	$[-0.0029 \quad 0.0089]$

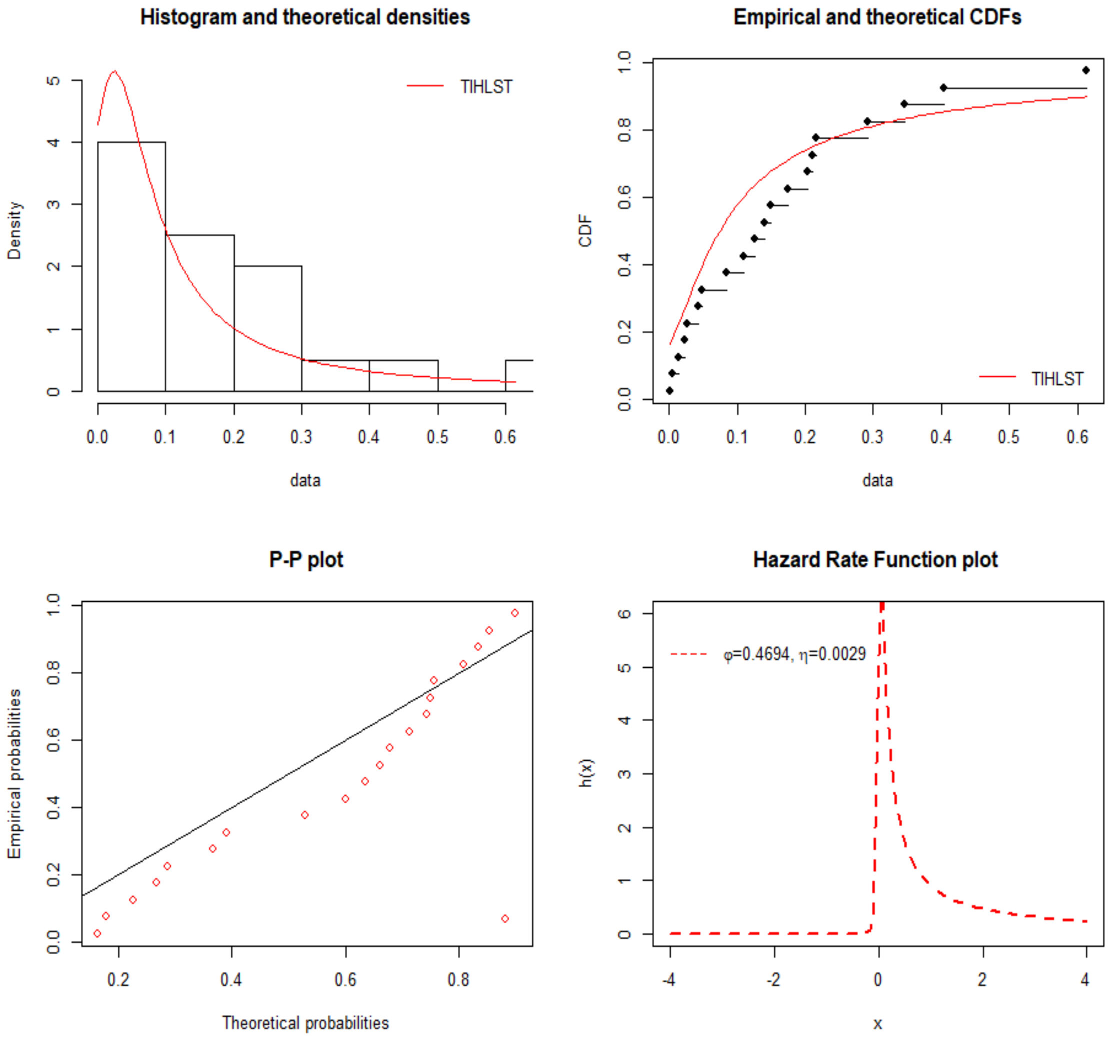

Fig. 3. Fitted density function plot *(top left panel)*, distribution function plot *(top right panel)*, probability-probability (PP) plot *(bottom left panel)* and hazard rate function plot *(bottom right panel)* of the $TIHL_{ST}$ distribution

6 Conclusion

This article presents a two-parameter distribution known as the type I half-logistic skew-t ($TIHL_{ST}$) distribution using the type I half-logistic transformation. The flexibility of the skew-t distribution is improved using this transformation. The structural properties such as the reliability analysis, failure rate function, reversed hazard rate function, cumulative hazard rate function, odds function, raw moment, quantile function, asymptotic behaviour, series expansion, probability weighted moments, order statistics and entropies of the $TIHL_{ST}$ distribution are derived. The type I half-logistic skew-t distribution parameter estimates were derived using the maximum likelihood estimation method and simulation studies carried-out to evaluate the finite sample performance of these parameter estimates showed that the parameter estimates were consistent and approached the true parameter values as the sample size is increased. More so, the application using a real dataset indicates that the $TIHL_{ST}$ distribution outperformed the other competing distributions and estimates of the parameters fall within the confidence intervals as indicated. In future research, the new $TIHL_{ST}$ distribution will be used as the distributed innovations distribution for the GARCH volatility modeling of financial return series. The research study will compare the performance of the $TIHL_{ST}$ distributed innovation to existing error innovations such as the normal distribution, Student-t distribution, generalized error distribution, and its skew variants in modeling and forecasting asset returns volatility.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

The asymptotic behaviour of the type-I half logistic skew-t ($TIHL_{ST}$) distribution is derived in full details. Firstly as $y \to -\infty$,

$$\lim_{y\to-\infty} f(y) = \lim_{y\to-\infty}\left(\frac{2\varphi\left(\dfrac{\eta}{2\left(\eta+x^2\right)^{3/2}}\right)\left[1-\left(\dfrac{1}{2}\left(1+\dfrac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi-1}}{\left\{1+\left[1-\left(\dfrac{1}{2}\left(1+\dfrac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi}\right\}^2}\right)$$

It is obvious that $\lim_{y\to-\infty}\left(\dfrac{\eta}{2\left(\eta+y^2\right)^{3/2}}\right)=0$. Hence,

$$\lim_{y\to-\infty} f(y) = 0\times\lim_{y\to-\infty}\left(2\varphi\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi-1}\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi}\right\}^{-2}\right)=0$$

Therefore, as $y \to -\infty$

$$\lim_{y\to-\infty} f(y)=0$$

Secondly as $y \to +\infty$,

$$\lim_{y\to+\infty} f(y) = \lim_{y\to+\infty}\left(\frac{2\varphi\left(\dfrac{\eta}{2\left(\eta+x^2\right)^{3/2}}\right)\left[1-\left(\dfrac{1}{2}\left(1+\dfrac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi-1}}{\left\{1+\left[1-\left(\dfrac{1}{2}\left(1+\dfrac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi}\right\}^2}\right)$$

It is obvious that $\lim_{y\to+\infty}\left(\dfrac{\eta}{2\left(\eta+y^2\right)^{3/2}}\right)=0$. Hence,

$$\lim_{y\to+\infty} f(y) = 0\times\lim_{y\to+\infty}\left(2\varphi\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi-1}\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+x^2}}\right)\right)\right]^{\varphi}\right\}^{-2}\right)=0$$

Therefore, as $y \rightarrow +\infty$

$$\lim_{y \rightarrow +\infty} f(y) = 0$$

This implies that the proposed TIHL_{ST} distribution has at least one mode.

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