

A New Improved Adomian Decomposition Method for Solving Emden–Fowler Type Equations of Higher–Order

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we present a suggested modification for Adomian decomposition method to solve Emden–Fowler Types Equations of higher-order ordinary differential equations. The proposed method can be applied to linear and non-linear problems. By using some illustrative examples, we tested the reliability and effectiveness of the proposed method and we found that the obtained results approximate the exact solution. Thus, we can conclude that this proposed method is efficient and reliable.

Keywords: Adomian decomposition method; boundary conditions; Emden–fowler equations.

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1 Introduction

We consider the Emden–Fowler Equations of the type

$$y' + \frac{m+1}{x}y + g(x, y) = 0, \quad (1.1)$$

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where $f(x)$ and $g(y)$ are known functions of x and y , respectively where $m \geq 0$, is called the shape factor. Emden-Fowler is one of the equations of great importance in mathematics and other sciences like fluid mechanics, quantum mechanics, chemical reactor hypothesis and geophysics. Emden-Fowler Equation has been the main interest of many researchers who investigated different types of Emden-Fowler Equation.

Recently, a number of researches have been solved Emden-Fowler Equation using a modified ADM [1-3].

The Adomian decomposition method is considered as one of the most effective methods in finding convergent as well as complete solution. In the year 1980s [4-7], the Adomian decomposition method (ADM) appeared by the American scientist Geoge. This method solved many equations that the traditional methods were unable to solve. Studied in this method [8-12] showed the efficiency and effectiveness of this method in finding approximate solution of different types of equations. Our goal in this work is to find approximate solutions for this type of equations using a modified method. We introduced a new operator that could solve like these equations.

2 Building Emden-Fowler Types Equations of Higher-Order

We suppose that the Emden-Fowler Eq.(1), was derived by using the equation

$$\frac{d}{dx}x^{-1-m} \frac{d}{dx}x^{1+m}(y) + g(x, y) = 0. \tag{2.1}$$

To conclude the Emden-Fowler types equations of different order, we employ the sense of Eq.(2) and set

$$\frac{d^n}{dx^n}x^{-1-m} \frac{d}{dx}x^{1+m}(y) + g(x, y) = 0, \tag{2.2}$$

where $n, m \geq 0$. To define such different order equations, we set n to different value i.e.

For $n = 0$ in Eq.(3), we have the first type

$$y' + \frac{(1+m)}{x}y + g(x, y) = 0.$$

For $n = 1$ in Eq.(3), we have the second type

$$y'' + \frac{(1+m)}{x}y' - \frac{(1+m)}{x^2}y + g(x, y) = 0.$$

For $n = 2$ in Eq.(3), we have the third type

$$y''' + \frac{(1+m)}{x}y'' - \frac{2(1+m)}{x^2}y' + \frac{2(1+m)}{x^3}y + g(x, y) = 0.$$

For $n = 3$ in Eq.(3), we have the fourth type

$$y'''' + \frac{(1+m)}{x}y''' - \frac{3(1+m)}{x^2}y'' + \frac{6(1+m)}{x^3}y' - \frac{6(1+m)}{x^4}y + g(x, y) = 0.$$

For $n = 4$ in Eq.(3), we have the fifth type

$$y''''' + \frac{(1+m)}{x}y'''' - \frac{4(1+m)}{x^2}y''' + \frac{12(1+m)}{x^3}y'' - \frac{24(1+m)}{x^4}y' + \frac{24(1+m)}{x^5}y + g(x, y) = 0,$$

...

sort equations of higher-order

$$y^{(n+1)} + \frac{(1+m)}{x}y^{(n)} - \frac{n(1+m)}{x^2}y^{(n-1)} + \sum_{k=2}^n \frac{(-1)^k n!(1+m)}{(n-k)!x^{k+1}}y^{(n-k)} + g(x, y) = 0, \quad (2.3)$$

3 Analysis of Method

To study the eq.(4), with

$$y(a) = b_0, y'(a) = b_1, \dots, y^{(n)}(a) = b,$$

where a, b_0, b_1, \dots, b are constants.

We rewrite eq.(4) as

$$Ly = -g(x, y), \quad (3.1)$$

where

$$L(.) = \frac{d^n}{dx^n}x^{(-1-m)}\frac{d}{dx}x^{(1+m)}(.),$$

and

$$L^{-1}(.) = x^{(-1-m)} \int_a^x x^{(1+m)} \underbrace{\int_a^x \int_a^x \dots \int_a^x}_{(n)} \underbrace{dx dx \dots dx}_{(n+1)}.$$

By applying L^{-1} on (5), we obtain

$$y(x) = \phi(x) - L^{-1}g(x, y), \quad (3.2)$$

where

$$L(\phi(x)) = 0.$$

The Adomain decomposition method represent the solution $y(x)$ and the non-linear function $g(x, y)$, by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (3.3)$$

and

$$g(x, y) = \sum_{n=0}^{\infty} A_n, \quad (3.4)$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined repeatable. By algorithm [4,6], the Adomain polynomials, which are obtain formula the following

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F \left(\sum_{i=0}^n \lambda^i y_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \dots,$$

which gives

$$\begin{aligned} A_0 &= F(y_0), \\ A_1 &= y_1 F'(y_0), \\ A_2 &= y_2 F'(y_0) + \frac{1}{2} y_1^2 F''(y_0), \\ A_3 &= y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F'''(y_0), \\ &\dots \end{aligned} \quad (3.5)$$

substituting Eq.(7) and Eq.(8) into Eq.(6), we get

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) - L^{-1} \sum_{n=0}^{\infty} A_n, \quad (3.6)$$

where, the components y_n can be specified as

$$\begin{aligned} y_0 &= \phi(x), \\ y_{n+1} &= L^{-1} A_n, \quad n \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} y_0 &= \phi(x), \\ y_1 &= L^{-1} A_0, \\ y_2 &= L^{-1} A_1, \\ y_3 &= L^{-1} A_2, \\ &\dots \end{aligned} \quad (3.7)$$

From (9) and (11), we find the components $y_n(x)$, and hence the series solution of $y(x)$ in (10) can be directly obtained.

For numerical aim, the nterm approximate

$$\Psi(x) = \sum_{k=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

4 Checkup of the Method and Applications of Numerical

In this section, we employ Modified Adomian Decomposition Method (MADM) to solving the Emden–Fowler types equations of different order and to get approximate solutions to exact solution.

4.1 Application 1

Consider the non-linear Emden–Fowler Type Equation at $n = 2, m = 0$ in Eq.(9),

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = y^2 - x^4, \quad (4.1)$$

$$y(1) = 1, y'(1) = 2, y''(1) = 2,$$

with exact solution $y = x^2$.

We rewrite Eq.(12) as

$$Ly = y^2 - x^4, \quad (4.2)$$

where

$$L(y) = \frac{d^2}{dx^2} x^{-1} \frac{d}{dx} x(y),$$

and

$$L^{-1}(\cdot) = x^{-1} \int_0^x x \int_1^x \int_1^x (\cdot) dx dx dx.$$

Applying L^{-1} to both sides on Eq.(13), we get

$$y(x) = x^2 + L^{-1}(y^2 - x^4), \tag{4.3}$$

To determine the components $y_n(x)$ for $y(x)$, into Eq.(14), we have

$$\sum_{n=0}^{\infty} y_n(x) = x^2 - L^{-1}(x^4) + L^{-1} \sum_{n=0}^1 A_n,$$

where non-linear y^2 , we have it as

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0. \end{aligned}$$

Employ Adomian decomposition method, the components $y_n(x)$, can be determined as

$$\begin{aligned} y_0 &= -0.0833333 x + 1.06667 x^2 - 0.00416667 x^7, \\ y_{n+1} &= L^{-1} A_n, n \geq 0, \end{aligned}$$

where A_n are Adomian polynomials that represent the nonlinear term $f(x, y)$, it gives

$$\begin{aligned} y_1 &= 0.0775363 x - 0.0615385 x^2 + 0.0000964506 x^5 - 0.00126984 x^6 + 0.00474074 x^7, \\ y_2 &= 0.00530791 x - 0.00466737 x^2 - 0.000179482 x^5 + \dots + 5.27022 \cdot 10^{-7} x^{10}. \end{aligned}$$

After computing the value of series components y_1, y_2 , second approximation of series solution is

$$\begin{aligned} y(x) = y_0 + y_1 + y_2 &= -0.000489111 x + 1.00046 x^2 - 0.0000830316 x^5 - 0.0000150754 x^6 + \\ &0.0000270656 x^7 - 2.87055 \cdot 10^{-8} x^9 + 5.27022 \cdot 10^{-7} x^{10}. \end{aligned}$$

Fig. 1. The exact solution and MADM solution.

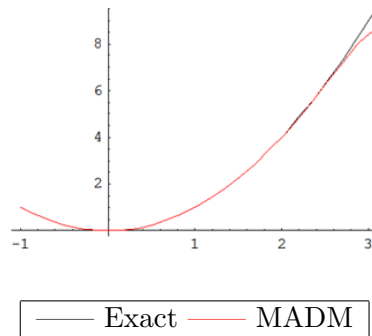


Fig. 1. The Approximation for the exact solution and MADM

4.2 Application 2

Consider the non-linear Emden–Fowler Type Equation

$$\begin{aligned} y' + \frac{3}{x}y &= 4 + \frac{3}{x} - (1+x)^3 + y^3, \\ y(0) &= 1, \end{aligned} \tag{4.4}$$

with exact solution $y = (x + 1)$.

By replace $m = 2, n = 0$ in Eq.(9), we have

$$Ly = 4 + \frac{3}{x} - (1 + x)^3 + y^3, \tag{4.5}$$

where

$$L(y) = x^{-3} \frac{d}{dx} x^3(y),$$

and

$$L^{-1}(\cdot) = x^{-3} \int_0^x x^3(\cdot) dx,$$

By applying L^{-1} on both sides of Eq.(16), we have

$$y(x) = L^{-1}(1 + 0.75x - 0.6x^2 - 0.5x^3 - 0.142857x^4) + L^{-1}(y^3). \tag{4.6}$$

To determine the components $y_n(x)$ for $y(x)$, into Eq.(17), we have

$$\sum_{n=0}^{\infty} y_n(x) = 1 + 0.75x - 0.6x^2 - 0.5x^3 - 0.142857x^4 + L^{-1} \sum_{n=0}^{\infty} A_n.$$

Employing MADM for y^3 , and the components $y_n(x)$, can be determined as

$$y_0 = 1 + 0.75x - 0.6x^2 - 0.5x^3 - 0.142857x^4,$$

$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

applying Adomain polynomial A_n , for the non-linear term y^3 , when for $n=0,1$, gives

$$A_0 = y_0^3,$$

$$A_1 = 3y_0^2y_1,$$

Thus, the solution is

$$y_1 = 0.25x + 0.45x^2 - 0.01875x^3 - 0.539732x^4 - 0.326384x^5 + \dots + 0.0258634x^{10},$$

$$y_2 = 0.15x^2 + 0.4125x^3 + 0.212946x^4 - 0.49865x^5 + \dots + 0.0473537x^{10},$$

The first terms, the approximate is following, respectively

$$y(x) = y_0 + y_1 + y_2 = 1 + 1. x + 2.22045 10^{-16} x^2 - 0.10625 x^3 - 0.469643 x^4 + \dots + 0.0732172 x^{10},$$

4.3 Application 3

If we put $n = 4, m = 1$ in Eq.(4), we obtain

$$y^{(5)} + \frac{2}{x}y^{(4)} - \frac{8}{x^2}y^{(3)} + \frac{24}{x^3}y^{(2)} - \frac{48}{x^4}y' + \frac{48}{x^5}y = \frac{e^x(48 - 48x + 24x^2 - 8x^3 + 2x^4 + x^5)}{x^5} - Ln(e^x + x) + Lny,$$

with

$$y(1), y'(1) = e^1 + 1, y''(1), y'''(1) = e^1,$$

the exact solution is $y = e^x + x$.

Table 1. The comparison of numerical errors between the exact solution and the "MADM"

x	Exact	MADN	Absolute Error
-0.3	0.7	0.70064	0.00064
-0.2	0.8	0.80032	0.00032
-0.1	0.9	0.90007	0.00007
0.0	1.0	1.00000	0.0000
0.1	1.1	1.09984	0.00016
0.2	1.2	1.19810	0.00019

Re-write in the above example as

$$Ly = \frac{e^x(48 - 48x + 24x^2 - 8x^3 + 2x^4 + x^5)}{x^5} - Ln(e^x + x) + Lny, \tag{4.7}$$

where

$$L^{-1}(\cdot) = x^{-2} \int_1^x x^2 \int_1^x \int_1^x \int_1^x (\cdot) dx dx dx dx.$$

Applying the L^{-1} , on Eq.(18), we have

$$y(x) = 4.62438x - 2.3785x^2 + 1.63097x^3 - 0.226523x^4 + L^{-1}\left(\frac{e^x(4848x + 24x^28x^3 + 2x^4 + x^5)}{x^5}\right)Ln(e^x + x) + Lny, \tag{4.8}$$

where, the components $y_n(x)$ for $y(x)$, into Eq. (19), we have

$$\sum_{n=0}^{\infty} y_n(x) = e^x + 4.62438x - 2.3785x^2 + 1.63097x^3 - 0.226523x^4 + 5.6229110^{-15}x(6.3504710^{14} + 4.011310^{14}x - \dots + 1.2647810^{10}x^{14}) + L^{-1} \sum_{n=0}^{\infty} A_n.$$

Employing MADM for Lny , and the components $y_n(x)$, can be determined as

$$y_0 = e^x + 4.62438x - 2.3785x^2 + 1.63097x^3 - 0.226523x^4 + 5.6229110^{-15}x(-6.3504710^{14} + 4.011310^{14}x - \dots + 1.2647810^{10}x^{14}),$$

$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

applying Adomain polynomial A_n , for the non-linear term Lny , when for $n = 0, 1$, gives

$$A_0 = Ly_0,$$

$$A_1 = \frac{y_1}{y_2}.$$

Then, we can compute the first few components respectively, as follows

$$y_1 = -0.157852x + 0.375144x^2 - 0.314662x^3 + 0.0889888x^4 + \dots + 0.000172769x^{15},$$

$$y_2 = \frac{5.4210110^{-20}(2.0237810^{16} - 5.9164810^{17}x^3 + \dots + 1.3559910^{15}x^{17})}{x^2} + 0.00595238x^5 \log(x),$$

Thus, the solution is

$$y(x) = y_0 + y_1 + y_2 = e^x + 4.46652x - 2.00335x^2 + 1.31631x^3 - 0.137535x^4 - 1.0269610^{-15}x^5 + 0.00213913x^6 - 0.000534423x^7 + 0.000282834x^8 + 0.000186658x^9 + 0.000146401x^{10} + \dots + 5.6229110^{-15}x(-6.3504710^{14} + 4.011310^{14}x - 2.7286310^{14}x^2 + \dots + 1.3559910^{15}x^{17}) + 0.00595238x^5 \log(x),$$

Fig. 2. The exact solution and the MADM solution.

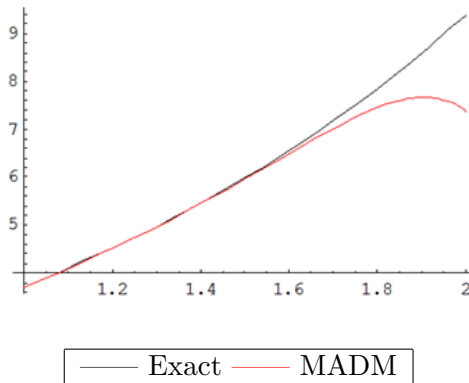


Fig. 2. The exact solution and the MADM solution

5 Conclusion

In this research, we attempt at introducing a new differential operator for solving various types of Emden-Fowler Equation of higher-order. It has been proved through many examples illustrated in this research that the solutions obtained by the MADM converge to the exact solution. Consequently, the introduced method is said to be very efficient for solving the equations considered.

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Competing Interests

Authors have declared that no competing interests exist.

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