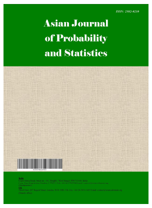


Asian Journal of Probability and Statistics

9(2): 18-31, 2020; Article no.AJPAS.61205

ISSN: 2582-0230

A Study of Some Properties and Goodness-of-Fit of a Gompertz-Rayleigh Model

Fadimatu Bawuro Mohammed[1], Kabiru Ahmed Manju[2], Umar Kabir Abdullahi[3], Makama Musa Sani[4] and Samson Kuje[5*]

[1]Department of Statistics, College of Science and Technology, Adamawa State Polytechnic, Yola, Adamawa State, Nigeria.
[2]Department of General Studies Education, School of General Education, Federal College of Education, Yola, Adamawa State, Nigeria.
[3]Department of Statistics, Ahmadu Bello University, P.M.B. 1045, Zaria, Kaduna State, Nigeria.
[4]Department of Statistics, Shehu Idris College of Health Science and Technology, Makarfi, Kaduna State, Nigeria.
[5]Department of Mathematical Science, Abubakar Tafawa Balewa University, Bauchi, Bauchi State, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. Authors FBM and KAM designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors UKA and MMS managed the analyses of the study. Author SK managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2020/v9i230223
Editor(s):
(1) Dr. Manuel Alberto M. Ferreira, Lisbon University, Portugal.
Reviewers:
(1) Miss Siddhika Arunachalam, University of Mumbai, India.
(2) Mudaser Ahad Bhat, Central University of Kashmir, India.
(3) Shiv Ram Meena, Guru Gobind Singh Indraprastha University, India.
Complete Peer review History: http://www.sdiarticle4.com/review-history/61205

Received: 07 July 2020
Accepted: 12 September 2020
Published: 03 October 2020

Original Research Article

Abstract

The Rayleigh was obtained from the amplitude of sound resulting from many important sources by Rayleigh. It is continuous probability distribution with a wide range of applications such as in life testing experiments, reliability analysis, applied statistics and clinical studies. However, it is not flexible enough for modeling heavily skewed datasets as compared to compound distributions. In this paper, we introduce a new extension of the Rayleigh distribution by using a Gompertz-G family of distributions. This paper defines and studies a three-parameter distribution called "Gompertz-Rayleigh distribution". Some properties of the proposed distribution are derived and discussed comprehensively in this paper and the

**Corresponding author: E-mail: kujesamson@gmail.com;*

three parameters are estimated using the method of maximum likelihood estimation. The goodness-of-fit of the proposed distribution is also evaluated by fitting it in comparison with some other existing distributions using a real life data.

Keywords: Rayleigh distribution; Gompertz-Rayleigh distribution; properties; parameters; method of maximum likelihood estimation and goodness-of-fit.

1 Introduction

Rayleigh [1] proposed a continuous distribution using the amplitude of sound resulting from different significant sources. This distribution has been applied in several areas such as in life testing experiments, reliability analysis, applied statistics and clinical studies. It is also a special case of the two-parameter Weibull distribution when the shape parameter becomes 2. The history and other useful features of the Rayleigh model could be seen in Siddiqui [2], Hirano [3] and Howlader and Hossian [4].

The probability density function (pdf) of the Rayleigh distribution with parameter θ is given by:

$$g(x) = \theta x e^{-\frac{\theta}{2}x^2} \tag{1}$$

And its cumulative distribution function (cdf) is given as

$$G(x) = 1 - e^{-\frac{\theta}{2}x^2} \tag{2}$$

for $x > 0, \theta > 0$ where Θ is the scale parameter.

Sequel to the simplicity and applicability of the Rayleigh distribution, there have been different developed generalized forms of the distribution leading to some capable models such as the generalized Rayleigh distribution (Kundu et al. [5]), Bivariate generalized Rayleigh distribution (Abdel-Hady [6]), Transmuted Rayleigh distribution (Merovci [7]), generalized Weibull-Rayleigh distribution (Yahaya and Alaku [8]), Weibull-Rayleigh distribution (Merovci and Elbatal [9]), transmuted Weibull-Rayleigh distribution (Yahaya and Ieren [10]) and Transmuted Inverse Rayleigh distribution (Ahmad et al. [11]) and so on.

Other generalized distributions away from Rayleigh distribution have shown that most compound distributions proposed in the literature fit various kind of data better than their standard counterparts. For instance, Weibull-Exponential distribution performed better than the Exponential distribution (Oguntunde et al. [12]), Weibull-Frechet distribution exhibited a very higher level of flexibility compared to the standard Frechet distribution (Afify et al. [13]), Lomax-Exponential distribution performed better than the exponential distribution (Ieren and Kuhe, [14]), others examples include Weibull-Lindley distribution (Ieren et al. [15]), Gompertz-Lindley distribution (Koleoso et al. [16]), Lomax-inverse Lindley distribution (Ieren et al. [17]), transmuted Lindley-Exponential distribution (Umar et al. [18]), Power Gompertz distribution (Ieren et al. [19]) and many others.

Therefore the proposed model would be a more flexible distribution with greater skewness and flexibility using the Gompertz-G family.

This article is presented in sections as follows. The definition of the new distribution and its plots are provided in section 2. Section 3 derived some Statistical properties of the new distribution. The estimation of unknown parameters of the proposed model using maximum likelihood estimation (MLE) is provided in section 4. In section 5, the proposed distribution with some competing distributions is applied to a real life dataset. Finally, a brief summary with some useful conclusions is given in section 5.

2 The Gompertz-Rayleigh Distribution (GomRD)

For any continuous distribution with cdf $G(x)$ and *pdf*, $g(x)$ Alizadeh et al. [20] proposed the Gompertz generalized (denoted "Gompertz-G) family of distributions that provides greater flexibility in modeling of real data sets. According to this methodology, the *cdf* and *pdf* of the Gompertz-G family are defined for any continuous distribution as:

$$F(x) = \int_{0}^{-\log[1-G(x)]} \alpha e^{\beta t} e^{-\frac{\alpha}{\beta}\left(e^{\beta t}-1\right)} dt \tag{3}$$

Solving the integral above, equation (3) yields the cdf and pdf of the Gompertz-G family as

$$F(x) = 1 - e^{\frac{\alpha}{\beta}\left\{1-\left[1-G(x)\right]^{-\beta}\right\}} \tag{4}$$

and

$$f(x) = \alpha g(x)\left[1-G(x)\right]^{-\beta-1} e^{\frac{\alpha}{\beta}\left\{1-\left[1-G(x)\right]^{-\beta}\right\}} \tag{5}$$

respectively, where g(x) and G(x) are the *pdf* and *cdf* of any continuous distribution to be generalized respectively and $\alpha > 0$ and $\beta > 0$ are the two additional shape parameters.

Making substituting of equation (1) and (2) in (4) and (5) above and simplifying, we obtain the *cdf* and *pdf* of the GomRD for a random variable X as:

$$F(x) = 1 - \exp\left\{\frac{\alpha}{\beta}\left[1-\left(1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right)^{-\beta}\right]\right\} \tag{6}$$

and

$$f(x) = \alpha\theta x e^{-\frac{\theta}{2}x^2}\left[1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right]^{-\beta-1} \exp\left\{\frac{\alpha}{\beta}\left[1-\left(1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right)^{-\beta}\right]\right\} \tag{7}$$

respectively, where $x > 0; \theta, \alpha, \beta > 0$.

The graph of the pdf and cdf of the GomRD using some arbitrary parameter values are shown as follows:

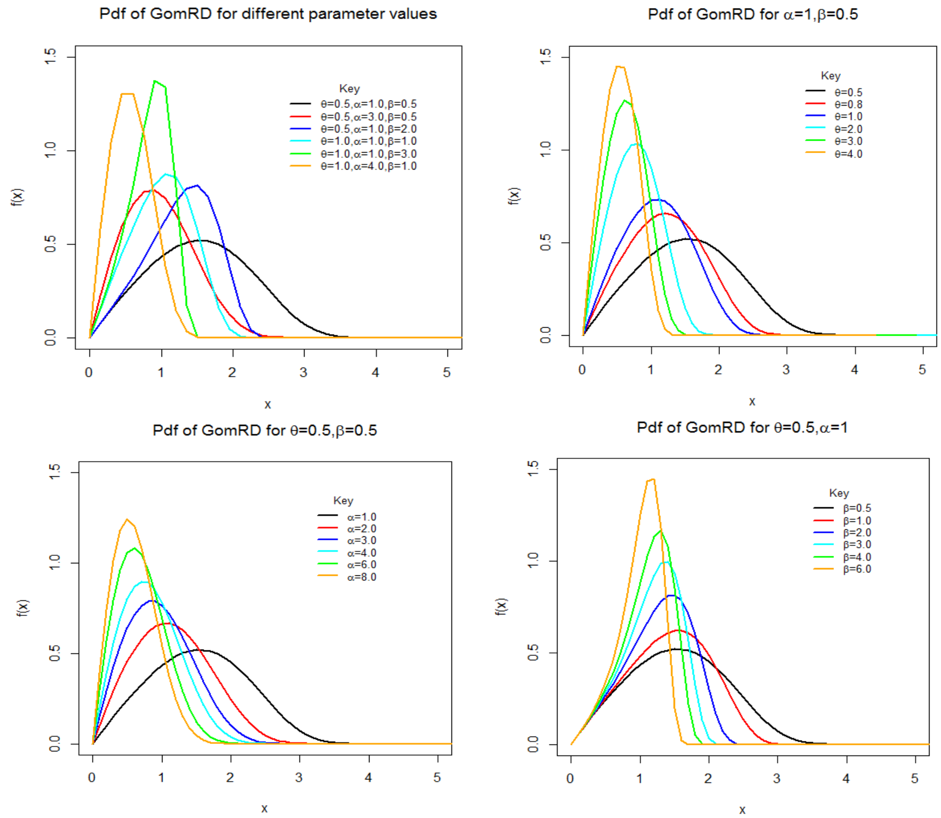
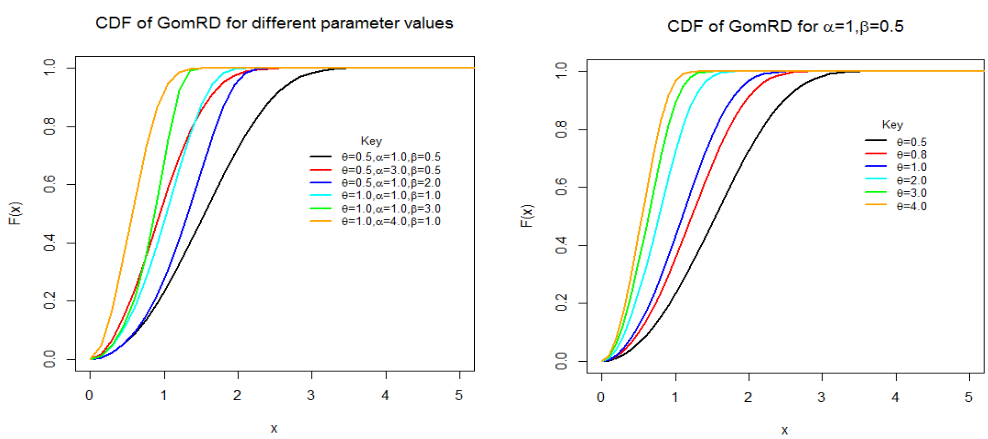


Fig. 1. PDF of Gom RD for selected parameter values

Fig. 1 revealed that the GomRD have different shapes and is right-skewed depending on values of the parameters. This means that distribution can be very useful for datasets with different shapes.



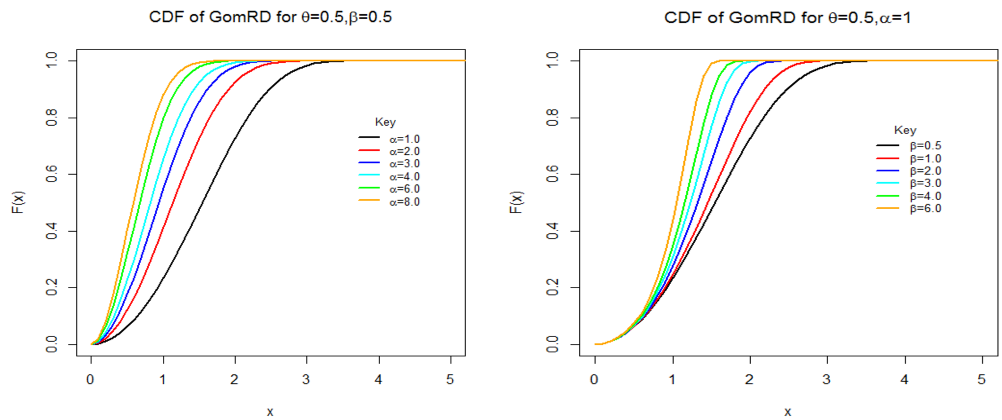


Fig. 2. CDF of GomRD for selected parameter values

From the above cdf plot, the cdf increases when x increases, and approaches 1 when x becomes large, as expected.

3 Statistical Properties of GomRD

3.1 Quantile function, median and simulation

Hyndman and Fan [21] defined the quantile function for any distribution in the form $Q(u)=F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0<u<1$

Using the cdf of the proposed model and applying simple algebra derive the quantile function of the GomRD as follows:

$$F(x)=1-\exp\left\{\tfrac{\alpha}{\beta}\left[1-\left(1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right)^{-\beta}\right]\right\}=u \tag{8}$$

Simplifying equation (8) above gives:

$$\left(1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right)^{-\beta}=1-\frac{\beta}{\alpha}\ln(1-u) \tag{9}$$

Collecting like terms in equation (9) and simplifying, the quantile function of the GomRD is obtained as:

$$Q(u)=\left\{\frac{2}{\theta\beta}\ln\left[1-\frac{\beta}{\alpha}\ln(1-u)\right]\right\}^{\frac{1}{2}} \tag{10}$$

where u is a uniform variate on the unit interval (0,1).

The median of X from the GomRD is simply obtained by setting u=0.5 and this substitution of $u = 0.5$ in Equation (10) which gives:

$$Median = \left\{ \frac{2}{\theta\beta} \ln\left[1 - \frac{\beta}{\alpha} \ln(0.5) \right] \right\}^{\frac{1}{2}} \tag{11}$$

Similarly, random numbers can be simulated from the GomRD by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\theta, \alpha, \beta > 0$ and $u \in (0,1)$:

$$X = \left\{ \frac{2}{\theta\beta} \ln\left[1 - \frac{\beta}{\alpha} \ln(1-u) \right] \right\}^{\frac{1}{2}} \tag{12}$$

"where u is a uniform variate on the unit interval (0,1).

Also, using equation (10), the quantile based measures of skewness and kurtosis are derived as follows:

Kennedy and Keeping [22] defined the Bowley's measure of skewness based on quartiles as:

$$SK = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{13}$$

And Moors [23] presented the Moors' kurtosis based on octiles by:

$$KT = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + \left(1/8\right)}{Q\left(6/8\right) - Q\left(1/8\right)} \tag{14}$$

"where $Q(.)$ is calculated by using the quantile function from equation (10).

3.2 Reliability analysis of the GomRD

This part of the article presents the survival (or reliability) function and the hazard (or failure) rate function of GomRD as follows:

The Survival function relates the likelihood of success for a unit or an individual after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{15}$$

Using the cdf of the GomRD in (15) and collecting like terms gives the survival function for the GomRD as:

$$S(x) = \exp\left\{ \frac{\alpha}{\beta}\left[1 - \left(1 - \left(1 - e^{-\frac{\theta}{2}x^2} \right) \right)^{-\beta} \right] \right\} \tag{16}$$

The graph of the survival function of the GomRD using different parameter values is shown below.

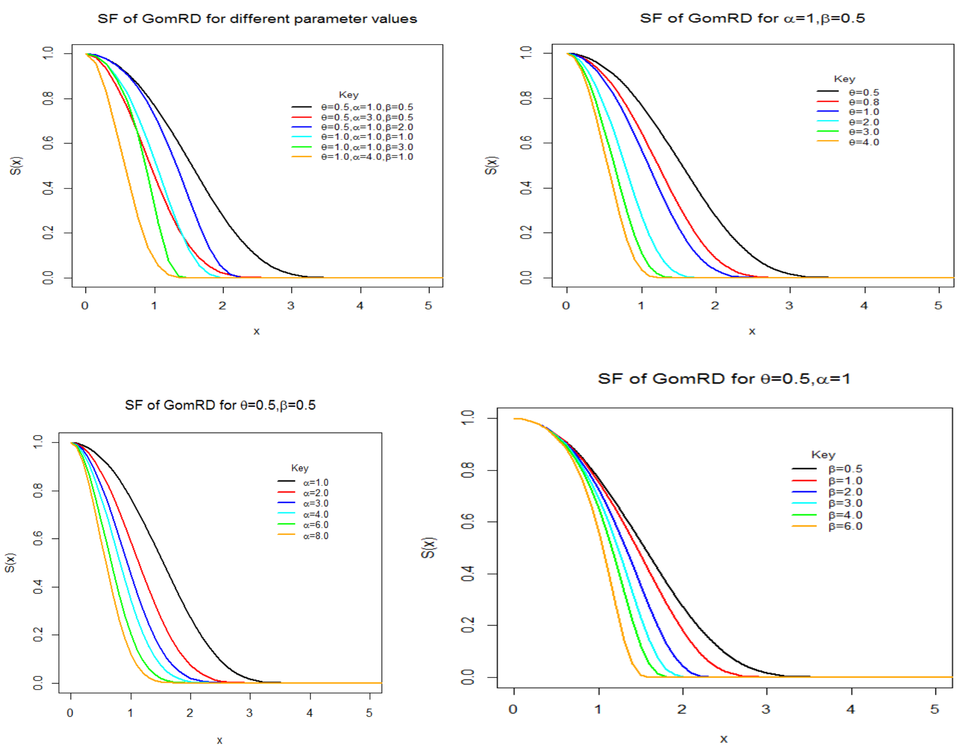


Fig. 3. Survival function of GomRD for selected parameter values

The graph of the survival function in Fig. 3 above reveals that the probability of survival for any random variable or event following a GomRD decreases as the variable or event advances in age or time, which means as time goes on, probability of life decreases. This implies that the GomRD could be used to analyze random variables whose survival rate decreases as they advance in age.

Hazard function is a mathematical function that relates the likelihood that a component will fail for an interval of time. The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{17}$$

Considering the pdf and cdf of GomRD, an expression for the hazard rate of GomRD is simplified and stated as:

$$h(x) = \alpha\theta x e^{-\frac{\theta}{2}x^2}\left[1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\right]^{-\beta-1} \quad (18)$$

where $x > 0, \theta, \alpha, \beta > 0$.

The graph of the hazard function for some parameter values is as given below:

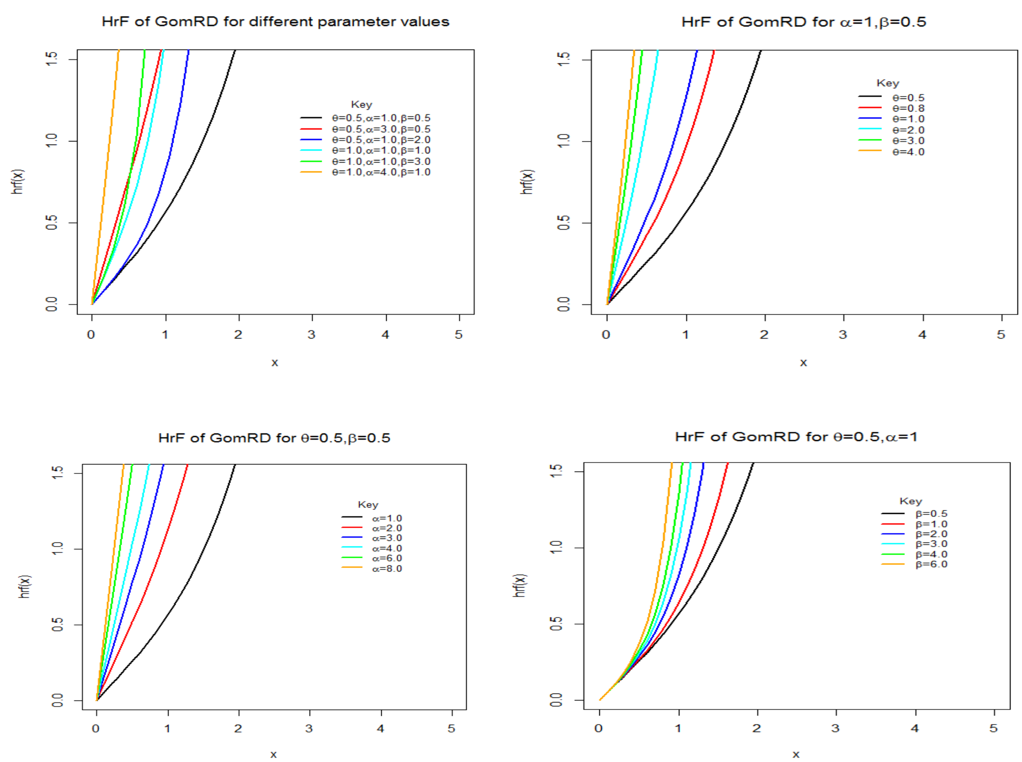


Fig. 4. Hazard function of GomRD for selected parameter values

Fig. 4 presents the graph of the hazard function of GomRD and it reveals that the failure rate for any random variable or event following a GomRD increases as the variable or event advance in time or age, meaning that as time grows, probability of death increases to infinity. This implies that the GomRD could be used to model random events whose failure rate increases per time or age.

4 Maximum Likelihood Estimation of the unknown Parameters of the GomRD

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the GomRD with unknown parameters, θ, α *and* β defined previously.

The likelihood function of the GomRD using the pdf in equation (7) is given by:

$$L\left(\underline{X}/\theta,\alpha,\beta\right)=\left(\alpha\theta\right)^{n}\prod_{i=1}^{n}\left(x_i^2\,\mathrm{e}^{-\frac{\theta}{2}x_i^2}\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]^{-\beta-1}\right)\exp\left\{\frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1-\left(1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right)^{-\beta}\right]\right\} \qquad (19)$$

Let the natural logarithm of the likelihood function be, $l\left(\kappa\right)=\log L\left(\underline{X}\mid\theta,\alpha,\beta\right)$, therefore, taking the natural logarithm of the function above gives:

$$l\left(\kappa\right)=n\log\alpha+n\log\theta+2\sum_{i=1}^{n}\log x_i-\frac{\theta}{2}\sum_{i=1}^{n}x_i^2-\left(\beta+1\right)\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]+\frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1-\left(1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right)^{-\beta}\right] \qquad (20)$$

Differentiating $l\left(\kappa\right)$ partially with respect to θ, α *and* β respectively gives the following results:

$$\frac{\partial l\left(\kappa\right)}{\partial\theta}=\frac{n}{\theta}-\frac{1}{2}\sum_{i=1}^{n}x_i^2+\frac{\left(\beta+1\right)}{2}\sum_{i=1}^{n}\left\{\frac{x_i^2\,\mathrm{e}^{-\frac{\theta}{2}x_i^2}}{\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]}\right\}-\frac{\alpha}{2}\sum_{i=1}^{n}\left\{x_i^2\,\mathrm{e}^{-\frac{\theta}{2}x_i^2}\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]^{-\beta-1}\right\} \qquad (21)$$

$$\frac{\partial l\left(\kappa\right)}{\partial\alpha}=\frac{n}{\alpha}+\frac{1}{\beta}\sum_{i=1}^{n}\left\{1-\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]^{-\beta}\right\} \qquad (22)$$

$$\frac{\partial l\left(\kappa\right)}{\partial\beta}=\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]^{-\beta}\ln\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right]\right\}-\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-\frac{\theta}{2}x_i^2}\right)\right] \qquad (23)$$

Making equation (21), (22) and (23) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters, $\hat{\theta}$, $\hat{\alpha}$ *and* $\hat{\beta}$. However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like *R*, *SAS*, *MATHEMATICA e.t.c.*

5 Application to a Real Life Dataset

In this section, an application to real life data is provided to illustrate the flexibility of the GomRD introduced in this study. We have compared the fitness of the GomRD to those of five other extensions of the Rayleigh model such as transmuted Weibull-Rayleigh distribution (TWRD), Weibull-Rayleigh distribution (WRD), transmuted Rayleigh distribution (TRD), Gompertz distribution (GomD) and the conventional Rayleigh distribution (RD). The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models.

The above listed distributions were fitted to the dataset and their performance was selected based on the following: Akaike Information Criterion, AIC, Consistent Akaike Information Criterion, CAIC, Bayesian Information Criterion, BIC, Hannan Quin Information Criterion, HQIC, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. More information about these information criteria can be found in Chen and Balakrishnan [24]. Note that the smaller these statistics are, the better the fit of the distribution.

Data set: This dataset has been used by Afify and Aryal [25], Barreto-Souza et al. [26], Bourguignon et al. [27], Oguntunde et al. [12], Ieren et al. [15], Ieren and Yahaya [28], Yahaya and Ieren [29], Koleoso et al. [16] and Smith and Naylor [30]. It is summarized as follows:

Table 1. Descriptive statistics of the dataset

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

Based on the statistics from Table 1, it is observed that the real life dataset is negatively skewed, and could be good for skewed models.

Table 2. Maximum likelihood parameter estimates for the dataset

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
GomRD	0.6056104	0.2787468	3.4678251	-
TWRD	0.06924352	6.26832073	1.14772063	0.84567528
WRD	0.0361874	5.0457529	0.5639456	
TRD	0.4081366	-	-	0.8893473
RD	0.8425334	-	-	-
GomD	-	0.1047741	1.9875491	-

Table 3. The statistics *ℓ*, AIC, CAIC, BIC and HQIC for the dataset

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
GomRD	21.46835	48.93671	49.34349	55.36611	51.46542	2nd
TWRD	67.72163	143.4433	144.1329	152.0158	146.8149	5th
WRD	89.70173	185.4035	185.8102	191.8329	187.9322	6th
TRD	51.06289	106.1258	106.3258	110.4121	107.8116	4th
RD	49.79089	101.5818	101.6474	103.7249	102.4247	3rd
GomD	29.00005	62.0001	62.2001	66.28637	63.68591	2nd

Table 4. The A^*, W^*, K-S statistic and P-values for the dataset

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
GomRD	1.07974	0.1770462	0.22822	0.002824	1st
TWRD	2.423038	0.4418065	0.29241	4.19e-05	5th
WRD	2.994072	0.5459647	0.34779	4.811e-07	6th
TRD	2.644015	0.4817336	0.30147	2.127e-05	4th
RD	2.553815	0.4654562	0.33392	1.584e-06	3rd
GomD	1.127883	0.2040339	0.32600	3.059e-06	2nd

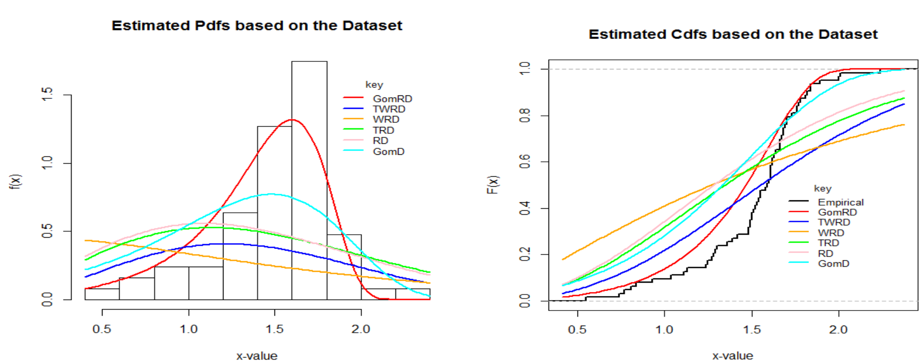


Fig. 5.1. Histogram and plots of the estimated densities and cdfs of the GomRD and other fitted distributions to the dataset

Tables 2 presents the values of the Maximum Likelihood Estimates of the model parameters for the real life dataset used, while table 3 presents the values of AIC, CAIC, BIC and HQIC for all the distributions fitted to the real life dataset and the values of A*, W* and K-S for the fitted distributions based on the dataset are provided in Tables 4.

The plots of the fitted density and cumulative distribution of the GomRD with those of competing distributions for the real life dataset are presented in Fig. 5.1. The PP-plots of the fitted distributions are also given in Figs. 5.2 based on the same dataset.

From the results in Tables 3 and 4 and with respect to all the model selection measures, it has been found that Gompertz-Rayleigh distribution (GomRD) has the lowest values of the measures compared to the other five fitted distributions and therefore it is considered as the best model to fit the dataset used. The results show that the GomRD is better than the five other fitted distributions (transmuted Weibull-Rayleigh distribution (TWRD), Weibull-Rayleigh distribution (WRD), transmuted Rayleigh distribution (TRD), Rayleigh distribution (RD) and the conventional Gompertz distribution (GomD)). These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in Figs. 5.1 and 5.2 respectively.

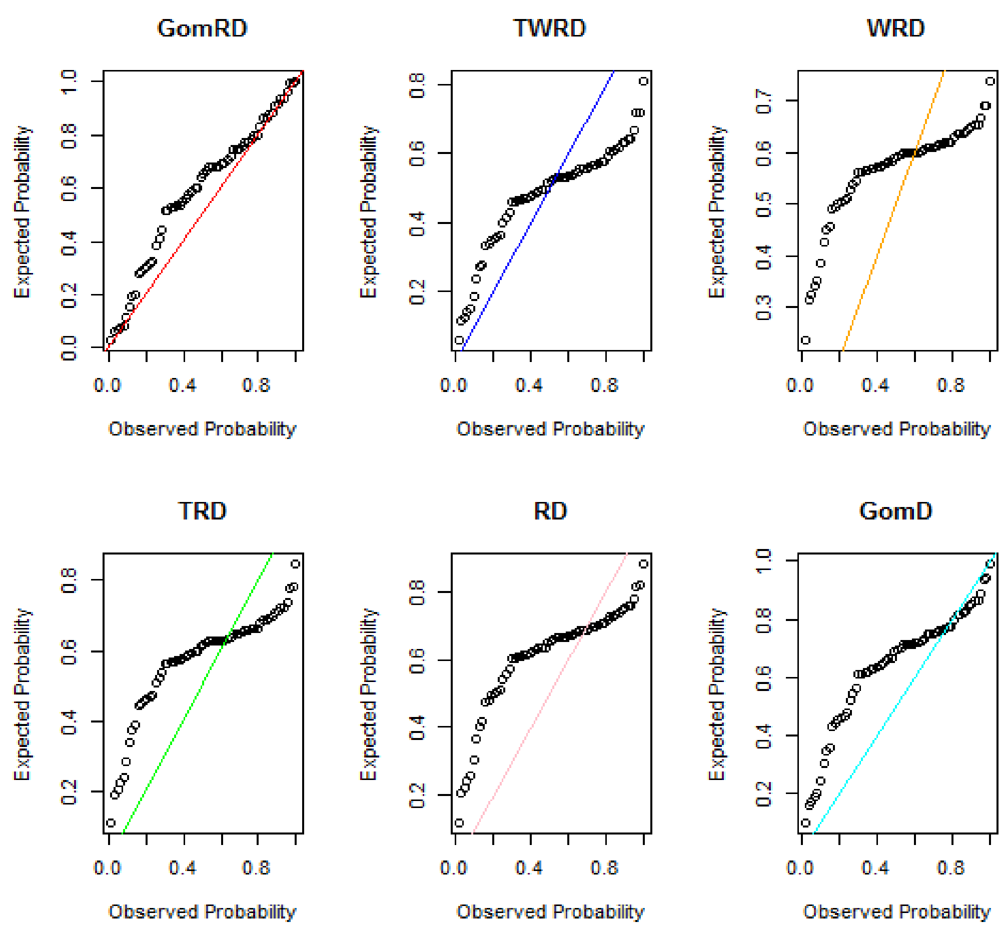


Fig. 5.2. Probability plots for the fit of the GomRD and other fitted models based on dataset

6 Summary and Conclusion

This paper has presented a three-parameter probability distribution called "a Gompertz-Rayleigh distribution". The paper has provided a comprehensive study of some statistical properties of the proposed distribution (Gompertz-Rayleigh) which include the derivation of explicit expressions for its survival function, hazard function and the quantile function which is useful for obtaining the median, skewness and kurtosis and simulation of random numbers from the proposed distribution. The estimation of the unknown model parameters using the method of maximum likelihood estimation has also being considered and treated in this study. The graph of the pdf of the proposed distribution has shown that its shape is skewed model and flexible for modeling real life data. Also, the graph of the survival and hazard functions are monotonically decreasing and increasing respectively. This means that the proposed model could be used for analyzing random events failure rate increase per age or time. It is also an indication that the proposed distribution could be a good choice for modeling cured fraction of surviving patients in a population. The performance of the proposed distribution (GomRD) has also been evaluated in this present paper using a real life dataset and the results revealed that the proposed distribution is much flexible compared to the other fitted distributions due to its ability to fit the real life dataset better than the other distributions as considered in this study. Based on this performance therefore we recommend that the proposed model be used for statistical quality control of variables whose distribution is skewed.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Rayleigh J. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. Phil. & Manag. 1980;10:73–78.

[2] Siddiqui MM. Some problems connected with Rayleigh distributions. Journal of Research Nat. Bureau of the Stand. 1962;60D:167–174.

[3] Hirano K. Rayleigh distributions. New York: John Wiley; 1986.

[4] Howlader HA, Hossain A. On Bayesian estimation and prediction from Rayleigh distribution based on type-II censored data. Comm. Stat.: Theo & Meth. 1995;24(9):2249-2259.

[5] Kundu D, Raqab MZ. Generalized Rayleigh distribution: Different methods of estimations. Comp. Stat. & Data Anal. 2005;49:187–200.

[6] Abdel-Hady DH. Bivariate generalized Rayleigh distribution. J. of Appl. Sci. Res. 2013;9(9):5403-5411.

[7] Merovci F. The transmuted Rayleigh distribution. Australian J. of Stat. 2013;22(1):21–30.

[8] Yahaya A, Alaku AY. On generalized Weibull-Rayleigh distribution: Its properties and applications. ATBU J. of Sci., Techn. and Edu. 2018;5(1):217-231.

[9] Merovci F, Elbatal I. Weibull Rayleigh distribution: Theory and applications. Appl. Math. & Inf. Sci. 2015;9(5):1-11.

[10] Yahaya A, Ieren TG. A note on the transmuted Weibull-Rayleigh distribution. Edited Proceedings of 1st Int. Conf. of Nigeria Stat. Soc. 2017;1:7-11.

[11] Ahmad A, Ahmad SP, Ahmed A. Transmuted inverse Rayleigh distribution: A generalization of the inverse Rayleigh distribution. Math. Theo. & Mod. 2015;4(7):90-98.

[12] Oguntunde PE, Balogun OS, Okagbue HI, Bishop SA. The Weibull-exponential distribution: Its properties and applications. J. of Appl. Sci. 2015;15(11):1305-1311.

[13] Afify MZ, Yousof HM, Cordeiro GM, Ortega EMM, Nofal ZM. The Weibull Frechet distribution and its applications. J. of Appl. Stat. 2016;1-22.

[14] Ieren TG, Kuhe AD. On the properties and applications of Lomax-exponential distribution. Asian J. of Prob. & Stat. 2018;1(4):1-13.

[15] Ieren TG, Oyamakin SO, Chukwu AU. Modeling lifetime data with Weibull-Lindley distribution. Biomet. & Biostat. Int. J. 2018;7(6):532–544.

[16] Koleoso PO, Chukwu AU, Bamiduro TA. A three-parameter Gompertz-Lindley distribution: Its properties and applications. J. of Math. Theo. & Mod. 2019;9(4):29-42.

[17] Ieren TG, Koleoso PO, Chama AF, Eraikhuemen IB, Yakubu N. A Lomax-inverse Lindley distribution: Model, properties and applications to lifetime data. J. of Adv. in Math. & Comp. Sci. 2019;34(3-4):1-28.

[18] Umar AA, Eraikhuemen IB, Koleoso PO, Joel J, Ieren TG. On the properties and applications of a transmuted Lindley-exponential distribution. Asian J. of Prob. & Stat. 2019;5(3):1-13.

[19] Ieren TG, Kromtit FM, Agbor BU, Eraikhuemen IB, Koleoso PO. A power Gompertz distribution: Model, properties and application to bladder cancer data. Asian Res. J. of Math. 2019;15(2):1-14.

[20] Alizadeh M, Cordeiro GM, Bastos Pinho LG, Ghosh I. The Gompertz-G family of distributions. J. of Stat. The and Pract. 2017;11(1):179–207.

[21] Hyndman RJ, Fan Y. Sample quantiles in statistical packages. The American Statistician. 1996;50(4): 361-365.

[22] Kenney JF, Keeping ES. Mathematics of statistics. 3 Edn, Chapman & Hall Ltd, New Jersey; 1962.

[23] Moors JJ. A quantile alternative for kurtosis. J. of the Royal Stat. Soc.: Series D. 1988;37:25–32.

[24] Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. Journal of Quality Technology. 1995;27:154–161.

[25] Afify AZ, Aryal G. The Kummaraswamy exponentiated Frechet distribution. J. of Data Sci. 2016;6:1-19.

[26] Barreto-Souza WM, Cordeiro GM, Simas AB. Some results for beta Frechet distribution. Comm. in Stat.: Theo. & Meth. 2011;40:798-811.

[27] Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. J. of Data Sci. 2014;12:53-68.

[28] Ieren TG, Yahaya A. The Weimal distribution: Its properties and applications. J. of the Nigeria Ass. of Math. Physics. 2017;39:135-148.

[29] Yahaya A, Ieren TG. On transmuted Weibull-exponential distribution: Its properties and applications. Nigerian J. of Sci. Res. 2017;16(3):289-297.

[30] Smith RL, Naylor JC. A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. J. of Appl. Stat. 1987;36:358-369.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/61205