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Equilibrium Points and Stability in R3BP with Heterogeneous Oblate Spheroid

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Authors' contributions

This work was carried out in collaboration between both authors. The first author formulated the problem and supervised the work, and also made corrections to the whole work while the second author carried out all mathematical calculations and numerical estimations of the work and prepared the manuscript.

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ABSTRACT

In this paper, equilibrium points and stability in the photogravitational restricted three-body problem (R3BP) with oblateness under a heterogeneous spheroid have been examined when the bigger primary is a radiating mass and the smaller one is a mass having three layers with different densities while the infinitesimal mass is an oblate spheroid. It is seen that for some values of oblateness of the infinitesimal mass, radiation pressure of the bigger primary, heterogeneity of the smaller and mass parameter μ, there exist up to five collinear equilibrium points all of which are unstable while a pair of triangular points exist and are stable when $0 < \mu < \mu_c$, where μ_c is the mass parameter defined by the radiation pressure, oblateness and heterogeneity.

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1. INTRODUCTION

A formulation which describes motion of a particle having infinitesimal mass and moving under the gravitational influence of two main bodies called primaries is a well known problem in celestial mechanics, galactic dynamics and other scientific fields. This formulation named by Lagrange [1] till date is called the restricted three-body problem (R3BP). There are five particular solutions, often referred to as equilibrium points; by virtue of their locations, three are called the collinear points and two are triangular points. The collinear points are unstable while the triangular solutions can be stable [2].

Over the years, the R3BP has gained relevance across the globe and several investigators have characterized the bodies by shapes, sizes and forms. Some have considered whether inclusion of some perturbing forces will alter the established results of the classical problem. Singh and Leke [3] investigated the stability of the photogravitational R3BP by characterizing the primaries as radiating bodies and having variable masses while the existence and stability of stationary solutions of the R3BP under the effect of the dissipative force, Stokes drag, have been carried out by Jain and Aggarwal [4] where they observed that there exist two non collinear stationary solutions which are unstable. Kumari et al. [5] examined existence and stability of triangular equilibrium points of the R3BP when the smaller primary is a heterogeneous triaxial rigid body while a study of five libration points in CR3BP under albedo effect was considered by Idrisi [6]. Ansari et al. [7] characterized the R3BP by taking the primaries as heterogeneous spheroids of three layers and the infinitesimal body varying its mass. The stability analysis of triangular equilibrium points in the R3BP when the primaries have Poynting-Robertson drag and are enclosed by circumbinary disc, has been studied by Singh and Amuda [8] while the libration points in the restricted three–body problem: euler angles, existence and stability was studied by Selim et al. [9]. Recently, Kumar and Sharma [10] studied effect of radiation pressure on resonant periodic orbits in photogravitational R3BP.

There have been few studies of the R3BP that took into account oblateness of the infinitesimal mass. Singh and Leke [11] have discussed motion in a modified Chermnykh's R3BP in which the infinitesimal mass is an oblate spheroid while Abouelmagd and Guirao [12] investigated analytically the perturbed planar R3BP in the case that the three involved bodies are oblate.

Most literature of celestial mechanics is full of a number of research papers in the restricted three-body problem, where primaries are either homogeneous spherical or non –spherical bodies. However, because some of the celestial bodies are stratified, e.g. our Earth is made up of three different layers. Therefore, in this paper, we carry out a survey of the equilibruim points and stability in the R3BP when the bigger primary is a radiating body and the smaller one is a heterogeneous oblate spheroid having three layers with different densities while the infinitesimal is an oblate spheroid. The structure of the paper is such that section 2 captures the dynamical equations of the model while section 3 and 4 examine position and stability of equilibrium points, respectively. Numerical illustrations are given in section 5 while the conclusion is drawn in section 6.

2. THE DYNAMICAL EQUATIONS

Let m_1, m_2 and m_3 be the masses of the bigger, smaller primary and the infinitesimal mass, respectively. We suppose that the bigger primary is a radiation emitter and the smaller one is a heterogeneous oblate spheroid with three layers of different densities $\rho_i\left(\rho_1 < \rho_2 < \rho_3\right)$ and axes $\left(a_i, b_i\right)$ $\left(i = 1,2,3\right)$, respectively; such that the equatorial plane coincides with the plane of motion. The infinitesimal mass is assumed to be an oblate spheroid which moves under the influence of both primaries. Let $O(x, y, z)$ be the Barycentric coordinate system with the line joining the primaries taken as the $x-$-axis, while the $y-$axis is perpendicular to it, and the $z-$axis is perpendicular to the orbital plane of the primaries. Let the coordinate system rotate with angular velocity n about the $z-$axis as seen in Fig. 1.

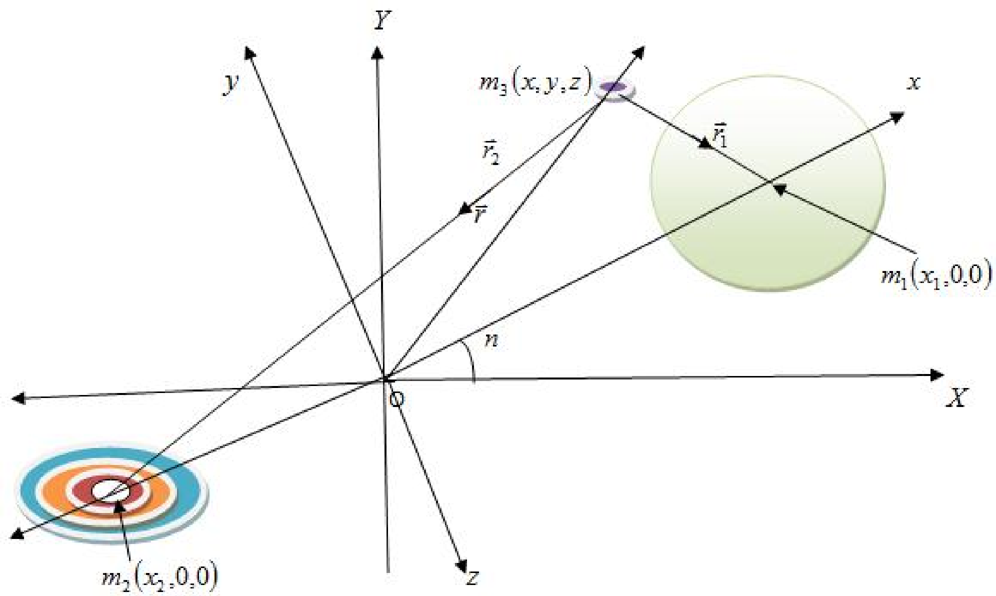

Fig. 1. Model of the R3BP with radiating bigger primary m_1, heterogeneous oblate small primary m_2 and oblate infinitesimal mass m_3

Following the methodology of Suraj et al. [13] and Singh and Leke [11], the equations of motion of the infinitesimal mass m_3 in the model of the CR3BP can be written, as

$$\ddot{x} - 2n\dot{y} = U_x$$

$$\ddot{y} + 2n\dot{x} = U_y \tag{1}$$

$$\ddot{z} = \Omega_z$$

where

$$U_x = n^2 x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3(x-\mu+1)k_2}{2r_2^5} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5} - \frac{3A_3\mu(x-\mu+1)}{2r_2^5}$$

$$U_y = \left[n^2 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_2}{2r_2^5} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5}\right]y$$

$$U_z = \left[-\frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_2}{2r_2^5} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5}\right]z$$

$$r_1^2 = (x-\mu)^2 + y^2 + z^2, \quad r_2^2 = (x-\mu+1)^2 + y^2 + z^2, \tag{2}$$

$$k_2 = \frac{4\pi}{3}\sum_{i=1}^{3}(\rho_i - \rho_{i+1})a_i b_i A_2 \;,\; A_2 = \frac{(a_i^2 - b_i^2)}{5R^2} \;,\; A_3 = \frac{(a^2 - b^2)}{5R^2}$$

$$n^2 = 1 + \frac{3}{2}k_3 \;,\; k_3 = \frac{4\pi}{3\mu}\sum_{i=1}^{3}(\rho_i - \rho_{i+1})a_i b_i A_2 \qquad (i = 1,2,3) \tag{3}$$

$$0 < \mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$$

$m_i(i = 1,2)$ are the masses of the bigger and smaller primary, respectively while n is the mean motion of the primaries. A_2 and A_3 are oblateness coefficient of the smaller primary and infinitesimal mass, respectively while k_2, k_3 appear because the smaller primary is a heterogeneous oblate spheroid and μ is the mass parameter while q is the radiation pressure factor of the bigger primary. $a_i, b_i(i = 1,2,3)$ are the axes of the three layers of the smaller primaries while a and b is the equatorial and polar radii of the infinitesimal mass, respectively.

3. LOCATIONS OF THE EQUILIBRIUM POINTS

The equilibrium points or particular solutions are the solutions of the dynamical system (1) when the infinitesimal mass is at rest. That is, we set $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = \ddot{z} = 0$ in equations (1) to get

$$n^2 x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3(x-\mu+1)k_2}{2r_2^5} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5} - \frac{3A_3\mu(x-\mu+1)}{2r_2^5} = 0$$

$$\left[n^2 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_2}{2r_2^5} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5}\right]y = 0 \tag{4}$$

$$\left[-\frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_2}{2r_2^5} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5}\right]z = 0$$

We shall consider motion and solutions in the $xy-$plane only.

3.1 Collinear Equilibrium Points

The collinear equilibrium points are the solutions of equations (4) when $y = z = 0$. That is we have to solve the first equation of (4). To do this we denote the equation by $f(x)$, so that

$$f(x) = n^2 x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3(x-\mu+1)k_2}{2r_2^5} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5} - \frac{3A_3\mu(x-\mu+1)}{2r_2^5} = 0 \tag{5}$$

where $r_1 = |x-\mu|$ and $r_2 = |x-\mu+1|$

The abscissas of the collinear points are the roots of equation (5).

Next, divide the orbital plane *Oxy* into three parts with respect to the primaries such that $x < \mu - 1$, $\mu - 1 < x < \mu$ and $\mu < x$. Therefore, when $x < \mu - 1$, implies that $x - \mu + 1 < 0$ and if this happens, then $x - \mu < 0$ and so we have from equation (5) that

$$n^2 x + \frac{q_1(1-\mu)(x-\mu)}{|x-\mu|^3} + \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} + \frac{3(x-\mu+1)k_2}{2|x-\mu+1|^5} + \frac{3A_3(1-\mu)(x-\mu)}{2|x-\mu|^5} + \frac{3A_3\mu(x-\mu+1)}{2|x-\mu+1|^5} = 0 \quad (6)$$

Also, when $\mu - 1 < x < \mu$, we have $x - \mu < 0$ and $x - \mu + 1 > 0$, and so we have

$$n^2 x + \frac{q(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} - \frac{3(x-\mu+1)k_2}{2|x-\mu+1|^5} + \frac{3A_3(1-\mu)(x-\mu)}{2|x-\mu|^5} - \frac{3A_3\mu(x-\mu+1)}{2|x-\mu+1|^5} = 0 \quad (7)$$

Finally, when $\mu < x$, we get

$$n^2 x - \frac{q(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} - \frac{3(x-\mu+1)k_2}{2|x-\mu+1|^5} - \frac{3A_3(1-\mu)(x-\mu)}{2|x-\mu|^5} - \frac{3A_3\mu(x-\mu+1)}{2|x-\mu+1|^5} = 0 \quad (8)$$

Hence, there exists a positive root for each of equations (6), (7) and (8) which give the locations of the collinear equilibrium points. We shall obtain them accordingly.

Collinear point L1

Now, in the first partition on the orbital plane, we let $x_1 = \mu - 1 - \varepsilon_1$ and substitute in (6), to get

$$n^2(\mu - 1 - \varepsilon_1 + \mu\varepsilon_1 - \mu\varepsilon_1) + q(1-\mu)(1+\varepsilon_1)^{-2} + \mu\varepsilon_1^{-2} + \frac{3}{2}k_2\varepsilon_1^{-4} + \frac{3}{2}A_3(1-\mu)(1+\varepsilon_1)^{-4}$$
$$+ \frac{3}{2}A_3\mu\varepsilon_1^{-4} = 0$$

Multiplying throughout by $-\frac{1}{n^2(1+\upsilon)}$ and simplifying, yields

$$\varepsilon_1^9 + \varepsilon_1^8\frac{(4\upsilon+5)}{1+\upsilon} + \varepsilon_1^7\frac{(6\upsilon+10)}{1+\upsilon} + \varepsilon_1^6\left[\frac{(4\upsilon+10)}{1+\upsilon} - \frac{(q+\upsilon)}{n^2(1+\upsilon)}\right] + \varepsilon_1^5\left[\frac{(\upsilon+5)}{1+\upsilon} - \frac{(2q+4\upsilon)}{n^2(1+\upsilon)}\right]$$
$$+ \varepsilon_1^4\left[\frac{1}{1+\upsilon} - \frac{(q+6\upsilon)}{n^2(1+\upsilon)} - \frac{3\upsilon}{2n^2(1+\upsilon)}\left(A_3 + \frac{k_2}{\mu}\right) - \frac{3A_3}{2n^2(1+\upsilon)}\right] - \frac{2\upsilon}{n^2(1+\upsilon)}\left[2 + 3\left(A_3 + \frac{k_2}{\mu}\right)\right]\varepsilon_1^3$$
$$- \frac{\upsilon}{n^2(1+\upsilon)}\left[1 + 9\left(A_3 + \frac{k_2}{\mu}\right)\right]\varepsilon_1^2 - \frac{6\upsilon}{n^2(1+\upsilon)}\left(A_3 + \frac{k_2}{\mu}\right)\varepsilon_1 - \frac{3\upsilon}{2n^2(1+\upsilon)}\left(A_3 + \frac{k_2}{\mu}\right) = 0$$

where $\upsilon = \frac{\mu}{(1-\mu)}$

Ignoring products of very small quantities, we get

$$\varepsilon_1^9 + a_1\varepsilon_1^8 + a_2\varepsilon_1^7 + a_3\varepsilon_1^6 + a_4\varepsilon_1^5 + a_5\varepsilon_1^4 - a_6\varepsilon_1^3 - a_7\varepsilon_1^2 - a_8\varepsilon_1 - a_9 = 0 \tag{9}$$

where $a_1 = \frac{(4\upsilon+5)}{1+\upsilon}$ $\quad a_2 = \frac{(6\upsilon+10)}{1+\upsilon}$ $\quad a_3 = \frac{1}{1+\upsilon}\left[9+3\upsilon+(1-q)+\frac{3}{2}(1+\upsilon)k_3\right]$

$a_4 = \frac{1}{1+\upsilon}\left[3-3\upsilon+2(1-q)+3(1+2\upsilon)k_3\right]$, $a_5 = \frac{1}{(1+\upsilon)}\left\{(1-q)-6\upsilon-\frac{3\upsilon k_2}{2\mu}+\frac{3}{2}(1+6\upsilon)k_3-\frac{3A_3}{2}\right\}$

$a_6 = \frac{2\upsilon}{(1+\upsilon)}\left[2+3\left(A_3+\frac{k_2}{\mu}-k_3\right)\right]$, $a_7 = \frac{\upsilon}{(1+\upsilon)}\left[1+9\left(A_3+\frac{k_2}{\mu}-\frac{1}{6}k_3\right)\right]$, $a_8 = \frac{6\upsilon(k_2+A_3\mu)}{\mu(1+\upsilon)}$,

$a_9 = \frac{3\upsilon(k_2+A_3\mu)}{2\mu(1+\upsilon)}$

Collinear point L_2

In this region, L_2 lies between the primaries and so, we substitute $x_2 = \mu - 1 + \varepsilon_2$ in equation (7). Following same methodology above, we get

$$\varepsilon_2^9 - b_1\varepsilon_2^8 + b_2\varepsilon_2^7 - b_3\varepsilon_2^6 + b_4\varepsilon_2^5 - b_5\varepsilon_2^4 - b_6\varepsilon_2^3 + b_7\varepsilon_2^2 - b_8\varepsilon_2 + b_9 = 0 \tag{10}$$

where

$b_1 = \frac{(4\upsilon+5)}{1+\upsilon}$, $\quad b_2 = \frac{(6\upsilon+10)}{1+\upsilon}$, $\quad b_3 = \frac{1}{1+\upsilon}\left[11+3\upsilon-(1-q)-\frac{3}{2}k_3(1+\upsilon)\right]$,

$b_4 = \frac{1}{1+\upsilon}\left[7-3\upsilon-2(1-q)-3k_3(1+2\upsilon)\right]$

$b_5 = \frac{1}{(1+\upsilon)}\left[2-6\upsilon-(1-q)-\frac{3\upsilon k_2}{2\mu}-\frac{3}{2}(1-6\upsilon)k_3+\frac{3}{2}(1-\upsilon)A_3\right]$

$b_6 = \frac{2\upsilon}{(1+\upsilon)}\left[2+3\left(\frac{k_2}{\mu}-k_3+A_3\right)\right]$, $b_7 = \frac{\upsilon}{(1+\upsilon)}\left[1+9\left(\frac{k_2}{\mu}-\frac{1}{6}k_3+A_3\right)\right]$

$b_8 = \frac{6\upsilon}{\mu(1+\upsilon)}(k_2+A_3\mu)$, $b_9 = \frac{3\upsilon(k_2+A_3\mu)}{2\mu(1+\upsilon)}$

Collinear point L_3 L_3 delete

For this point, since it lies to the right of the primaries, we substitute $x_3 = \mu + \varepsilon_3$ in equation (8). Solving, we get

$$\varepsilon_3^9 + c_1\varepsilon_3^8 + c_2\varepsilon_3^7 + c_3\varepsilon_3^6 + c_4\varepsilon_3^5 - c_5\varepsilon_3^4 - c_6\varepsilon_3^3 - c_7\varepsilon_3^2 - c_8\varepsilon_3 - c_9 = 0 \tag{11}$$

where

$c_1 = \frac{(5\upsilon + 4)}{(1+\upsilon)}$, $c_2 = \frac{(10\upsilon + 6)}{(1+\upsilon)}$, $c_3 = \frac{1}{(1+\upsilon)}\left[3 + 9\upsilon + (1-q) + \frac{3}{2}(1+\upsilon)k_3\right]$,

$$c_4 = \frac{1}{(1+\upsilon)}\left[3\upsilon + (1-q) + \frac{3}{2}(1+\upsilon)k_3\right],$$

$$c_5 = \frac{1}{(1+\upsilon)}\left[4 - 4(1-q) + \frac{3\upsilon k_2}{2\mu} - \frac{3(4+\upsilon)}{2}k_3 + \frac{3(1+\upsilon)A_3}{2}\right]$$

$c_6 = \frac{6}{(1+\upsilon)}\left[1-(1-q) + A_3 - \frac{3}{2}k_3\right]$, $c_7 = \frac{4}{(1+\upsilon)}\left[1-(1-q) + \frac{9}{4}A_3 - \frac{3}{2}k_3\right]$,

$c_8 = \frac{6A_3}{(1+\upsilon)}$, $c_9 = \frac{A_3}{(1+\upsilon)}$

We shall explore the polynomials (9), (10) and (11) numerically to determine the roots $\varepsilon_i\,(i = 1,2,3)$ corresponding to the collinear equilibrium points $L_i\,(i = 1,2,3)$.

3.2 Triangular Equilibrium Points

The triangular equilibrium points are the solutions of equations (4) when $z = 0$. Therefore, we have to solve equations

$$n^2 x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3(x-\mu+1)k_2}{2r_2^5} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5} - \frac{3A_3\mu(x-\mu+1)}{2r_2^5} = 0$$

and

$$n^2 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_2}{2r_2^5} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5} = 0 \qquad (12)$$

Re-writing first equation of (3) and substituting the second in it, our result yields the equations

$$n^2 - \frac{q}{r_1^3} - \frac{3A_3}{r_1^5} = 0,\ (1-\mu) \neq 0 \qquad (13)$$

and

$$\mu\left(n^2 - \frac{1}{r_2^3} - \frac{3A_3}{r_2^5}\right) = \frac{3k_2}{2r_2^5} \qquad (14)$$

Now, if all the imposed perturbing forces are relaxed such that $k_{2,3} = 0$, $q_1 = 1$ and $A_3 = 0$; equations (13) and (14) reduce to the classical case of Szebehely [2], where $r_1 = r_2 = 1$.

Since our model modifies the classical case of the R3BP, we assume that the modified solutions can be expressed as

$$r_i = 1 + \chi_i \tag{15}$$

where $|\chi_i| << 1 \ (i = 1,2)$

From equations (13), if only linear terms in $\chi_i (i = 1,2)$ are retained, we get the following relations:

$$r_i^{-3} = 1 - 3\chi_i \,, r_i^{-5} = 1 - 5\chi_i \tag{16}$$

Substituting equations (3) and (16) in equations (13) and (14), respectively, we get

$$\chi_1 = -\frac{1}{3}(1-q) - \frac{1}{2}k_3 + \frac{1}{2}A_3 \tag{17}$$

$$\chi_2 = \frac{1}{2}\frac{k_2}{\mu} - \frac{1}{2}k_3 + \frac{1}{2}A_3$$

Therefore, we have

$$r_1 = 1 - \frac{1}{3}(1-q) - \frac{1}{2}k_3 + \frac{1}{2}A_3 \tag{18}$$

$$r_2 = 1 + \frac{1}{2}\frac{k_2}{\mu} - \frac{1}{2}k_3 + \frac{1}{2}A_3$$

We have obtained equations (18) by retaining only linear terms and neglecting second and higher order terms and products of the strictly small factors.

Finally, solving equations (2) when $z = 0$, we obtain the coordinates of the triangular points as

$$x = \mu - \frac{1}{2} + \frac{r_2^2 - r_1^2}{2} \quad \text{and}$$

$$y = \pm\frac{1}{2}\sqrt{2\left(r_1^2 + r_2^2\right) - 1 - \left(r_2^2 - r_1^2\right)^2}$$

Substituting the distances (18) in these equations and simplifying, yields the solutions

$$x = \mu - \frac{1}{2} + \frac{1}{3}(1-q) + \frac{k_2}{2\mu} \tag{19}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{9}(1-q) - \frac{1}{3\mu}k_2 + \frac{2}{3}(k_3 + A_3)\right]$$

These solutions are denoted by L_4 and L_5 and form two triangles with the lines joining the primaries. Clearly, the points are characterized by the mass parameter, radiation pressure of the bigger primary, heterogeneity of the smaller and oblateness of the infinitesimal mass.

4. STABILITY OF EQUILIBRIUM POINTS

In the study of the stability of equilibrium points in the R3BP, usually, for all values of the free parameters, the equilibrium points may be stable or otherwise. What is required is to take small deviations from the equilibrium points by displacing the infinitesimal mass a little from rest. Once the infinitesimal mass oscillates about the equilibrium point, then the point is a stable equilibrium point. If however, the infinitesimal escapes from the neighborhood of the equilibrium point, then the point is an unstable location.

Now, let $x = x_0 + \xi$, $y = y_0 + \eta$, where (ξ, η) is a small displacement from the equilibrium point (x_0, y_0). Consequently, the variational equations of motion and the corresponding characteristic equation are respectively:

$$\ddot{\xi} - 2n\dot{\eta} = U_{xx}^0 \xi + U_{xy}^0 \eta$$

$$\ddot{\eta} + 2n\dot{\xi} = U_{xy}^0 \xi + U_{yy}^0 \eta \tag{20}$$

and

$$\lambda^4 - \left(U_{xx}^0 + U_{yy}^0 - 4n^2\right)\lambda^2 + U_{xx}^0 U_{yy}^0 - \left(U_{xy}^0\right)^2 = 0 \tag{21}$$

where, U_{xx}^0 , U_{yy}^0 and U_{xy}^0 are second order partial derivatives which will be evaluated at the equilibrium points.

Now, we can investigate the stability of the equilibrium points

4.1 Collinear Equilibrium Points

We consider motion around the collinear point L_1. In this case the partial derivatives estimated at this point are

$$U_{xx}^{0}=1+\frac{2(1-\mu)q}{\left|x_0-\mu\right|^3}+\frac{2\mu}{\left|x_0-\mu+1\right|^3}+\frac{6k_2}{\left|x_0-\mu+1\right|^5}+\frac{3}{2}k_3+\frac{6(1-\mu)A_3}{\left|x_0-\mu\right|^5}+\frac{6\mu A_3}{\left|x_0-\mu+1\right|^5}$$

$$U_{yy}^{0}=1-\frac{q(1-\mu)}{\left|x_0-\mu\right|^3}-\frac{\mu}{\left|x_0-\mu+1\right|^3}-\frac{3k_2}{2\left|x_0-\mu+1\right|^5}+\frac{3}{2}k_3-\frac{3(1-\mu)A_3}{2\left|x_0-\mu\right|^5}-\frac{3\mu A_3}{2\left|x_0-\mu+1\right|^5} \tag{22}$$

$$U_{xy}^{0}=0$$

The second order partial derivatives have been estimated at the collinear point $(x_0,0,0)$.

Consequently,

$$U_{xx}^{0}+U_{yy}^{0}-4n^2=-2+\frac{q(1-\mu)}{\left|x_0-\mu\right|^3}+\frac{\mu}{\left|x_0-\mu+1\right|^3}+\frac{9k_2}{2\left|x_0-\mu+1\right|^5}-3k_3+\frac{9(1-\mu)A_3}{2\left|x_0-\mu\right|^5}+\frac{9\mu A_3}{2\left|x_0-\mu+1\right|^5}$$

$$\begin{aligned}U_{xx}^{0}U_{yy}^{0}={}&1+\frac{(1-\mu)}{\left|x_0-\mu\right|^3}+\frac{\mu}{\left|x_0-\mu+1\right|^3}-\frac{(1-\mu)(1-q)}{\left|x_0-\mu\right|^3}+\frac{9k_2}{2\left|x_0-\mu+1\right|^5}+\frac{9(1-\mu)A_3}{2\left|x_0-\mu\right|^5}+\frac{9\mu A_3}{2\left|x_0-\mu+1\right|^5}\\&+2k_3+\frac{2(1-\mu)k_3}{\left|x_0-\mu\right|^3}+\frac{2\mu k_3}{\left|x_0-\mu+1\right|^3}-\frac{2(1-\mu)^2}{(x_0-\mu)^6}-\frac{2\mu^2}{(x_0-\mu+1)^6}+\frac{4(1-\mu)^2(1-q)}{(x_0-\mu)^6}-\frac{9\mu k_2}{(x_0-\mu+1)^6}\\&-\frac{3(1-\mu)A_3}{(x_0-\mu)^6}-\frac{3\mu A_3}{(x_0-\mu+1)^6}-\frac{4\mu(1-\mu)}{\left|(x_0-\mu)(x_0-\mu+1)\right|^3}+\frac{4\mu(1-\mu)(1-q)}{\left|(x_0-\mu)(x_0-\mu+1)\right|^3}-\frac{9(1-\mu)k_2}{\left|(x_0-\mu)(x_0-\mu+1)\right|^3}\\&-\frac{6(1-\mu)^2A_3}{(x_0-\mu)^6}-\frac{6\mu^2A_3}{(x_0-\mu+1)^6}-\frac{18\mu(1-\mu)A_3}{\left|(x_0-\mu)(x_0-\mu+1)\right|^3}\end{aligned}$$

Hence, the characteristic equation in the case of the collinear equilibrium point is

$$\lambda^4-a_1\lambda^2+a_2=0 \tag{23}$$

where

$$a_1=-2\left[1-\frac{(1-\mu)}{2\left|x_0-\mu\right|^3}+\frac{(1-\mu)(1-q)}{2\left|x_0-\mu\right|^3}-\frac{\mu}{2\left|x_0-\mu+1\right|^3}-\frac{9k_2}{4\left|x_0-\mu+1\right|^5}+\frac{3}{2}k_3-\frac{9(1-\mu)A_3}{4\left|x_0-\mu\right|^5}\right.$$
$$\left.-\frac{9\mu A_3}{4\left|x_0-\mu+1\right|^5}\right]$$

$$a_2 = 1 + \frac{(1-\mu)}{|x_0-\mu|^3}\left[1 - \frac{2(1-\mu)}{|x_0-\mu|^3} - \frac{4\mu}{|(x_0-\mu+1)|^3}\right] + \frac{\mu}{|x_0-\mu+1|^3}\left[1 - \frac{2\mu}{|(x_0-\mu+1)|^3}\right]$$

$$-\frac{(1-\mu)}{|x_0-\mu|^3}\left[1 - \frac{4(1-\mu)}{(x_0-\mu)^2} - \frac{4\mu}{|(x_0-\mu+1)|^3}\right](1-q) + \frac{9}{2}\left[\frac{1}{|x_0-\mu+1|^5} - \frac{2\mu}{(x_0-\mu+1)^6}\right.$$

$$\left. - \frac{2(1-\mu)}{|(x_0-\mu)(x_0-\mu+1)|^3}\right]k_2 + 2\left[1 + \frac{(1-\mu)}{|x_0-\mu|^3} + \frac{\mu}{|x_0-\mu+1|^3}\right]k_3 - 3\left[\frac{(1-\mu)}{(x_0-\mu)^6} + \frac{\mu}{(x_0-\mu+1)^6}\right.$$

$$\left. + \frac{2(1-\mu)^2}{(x_0-\mu)^6} + \frac{2\mu^2}{(x_0-\mu+1)^6} + \frac{6\mu(1-\mu)}{|(x_0-\mu)(x_0-\mu+1)|^3} - \frac{3(1-\mu)}{2|x_0-\mu|^5} - \frac{3\mu}{2|x_0-\mu+1|^5}\right]A_3$$

Now, the stability outcome will depend on the types of roots of the characteristic equation (23) and the nature of the roots will be defined by the coefficients $a_i\,(i=1,2)$. Hence, we shall numerically compute these roots in a later section of the paper and then come up with a result that will tell whether motion around the collinear equilibrium points is stable or unstable

4.2 Triangular Equilibrium Points

The stability of the triangular points is equally determined by the nature of the roots of the characteristic equation (21). To obtain the derivatives computed in this case, we first compute the second order partials with the help of equation (12), to get:

$$U_{xx} = 3\left[\frac{q(1-\mu)(x-\mu)^2}{r_1^5} + \frac{\mu(x-\mu+1)^2}{r_2^5} + \frac{5(x-\mu+1)^2 k_2}{2r_2^7} + \frac{5}{2}\frac{(1-\mu)(x-\mu)^2 A_3}{r_1^7} + \frac{5}{2}\frac{\mu(x-\mu+1)^2 A_3}{r_1^7}\right]$$

$$U_{yy} = 3y^2\left[\frac{q(1-\mu)}{r_1^5} + \frac{\mu}{r_2^5} + \frac{5}{2r_2^7}k_2 + \frac{5}{2r_1^7}(1-\mu)A_3 + \frac{5}{2r_2^7}\mu A_3\right]$$

$$U_{xy} = 3y\left[\frac{q(1-\mu)(x-\mu)}{r_1^5} + \frac{\mu(x-\mu+1)}{r_2^5} + \frac{5(x-\mu+1)k_2}{2r_2^7} + \frac{5(1-\mu)(x-\mu)A_3}{2r_1^7} + \frac{5\mu(x-\mu+1)A_3}{2r_1^7}\right]$$

Now, using equations (18) and (19), we ignore products and retain only linear terms of k_i, $1-q$ and A_3, to get the second order partial derivatives computed at the triangular points:

$$U^0_{xx} = \frac{3}{4}\left[1 - \frac{2}{3}(1-q) + 2\mu(1-q) + 4\left(1 - \frac{1}{2\mu}\right)k_2 + \frac{5}{2}k_3\right]$$

$$U^0_{yy} = \frac{9}{4}\left\{1 + \frac{2}{9}(1-q) - \frac{2}{3}\mu(1-q) + \frac{2}{3\mu}k_2 + \frac{7}{6}k_3 + \frac{4}{3}A_3\right\} \tag{24}$$

$$U^0_{xy} = \sqrt{3}\left[-\frac{3}{4} + \frac{1}{6}p_1 + \frac{1}{2}k_2 - \frac{11}{8}k_3 - \frac{1}{2}A_3 + \mu\left\{\frac{3}{2} + \frac{1}{6}p_1 + \frac{11}{4}k_3 + A_3 + \frac{k_2}{2\mu^2}\right\}\right]$$

Now, substituting equations (24) in the characteristic equation (21) and simplifying, we get

$$\lambda^4 + \left(1 - 3k_2 + \frac{3}{2}k_3 - 3A_3\right)\lambda^2 + \frac{3\mu(1-\mu)}{4}\left[9 + 12A_3 + 2(1-q) + \frac{6}{\mu}k_2 + 33k_3\right] = 0 \quad (25)$$

The roots of (21) are

$$\lambda_{1,2,3,4} = \pm\omega_i (i = 1,2) \quad (26)$$

where $\omega_i = \dfrac{-b \pm \sqrt{\Delta}}{2}$

$$\Delta = 3[9 + 2(1-q) + 33k_3 + 12A_3]\mu^2 - 3(9 + 2(1-q) - 6k_2 + 33k_3 + 12A_3)\mu + 1 - 24k_2 + 3k_3 - 6A_3$$

Δ is the discriminant and is a strictly decreasing function of μ in the interval $\left(0, \frac{1}{2}\right)$ and has values of opposite signs at the end points. Therefore, there exists a value of μ , at which the discriminant is zero. This value is called the critical mass parameter and it is given by

$$\mu_C = \frac{1}{2}\left(1 - \frac{\sqrt{23}}{3\sqrt{3}}\right) - \frac{2(1-q)}{27\sqrt{69}} - \frac{1}{3}\left[\left(1 + 5\sqrt{\frac{3}{23}}\right)k_2 + \frac{2}{3\sqrt{69}}k_3\right] - \frac{22A_3}{9\sqrt{69}} \quad (27)$$

The first term is the Routh's critical mass value for the classical case and the second term is the presence of radiation pressure of the bigger primary while the third and last are the effect arising from heterogeneity of the smaller primary and the fact that the infinitesimal mass is an oblate spheroid, respectively. Evidently, equation (27) is a decreasing function of the parameters q , k_2 , k_3 and A_3 .

Next, we analyze the stability outcome which depends on the nature of the roots (26) and the kinds of roots in (26) will depend on the discriminant Δ. Thus we consider the nature of the discriminant based on the relation between the mass ratio μ and the critical mass parameter μ_C . The roots and stability results are as follows:

i. When $0 < \mu < \mu_C$ the discriminant Δ is positive and the roots (26) are all distinct pure imaginary. In this case the triangular equilibrium points are stable and motion is bounded and defined by two oscillatory solutions which can be written following Szebehely [2]:

$$\xi = A_1 \cos\omega_1 t + A_2 \sin\omega_1 t + A_4 \cos\omega_2 t + A_5 \sin\omega_2 t$$

$$\eta = B_1 \cos\omega_1 t + B_2 \sin\omega_1 t + B_4 \cos\omega_2 t + B_5 \sin\omega_2 t$$

where $A_{1,2}, B_{1,2}$ and $A_{4,5}, B_{4,5}$ are the long and short periodic terms

ii. When $\mu = \mu_C$, $\Delta = 0$; two roots are equal and the triangular equilibrium points are unstable.

iii. When $\mu_C < \mu$ the discriminant is negative, and the roots are complex with two having positive real parts. This induces instability and the equilibrium points are unstable.

5. NUMERICAL ILLUSTRATIONS

We have analytically computed the polynomials for the positions of the collinear equilibrium points. Each of the polynomial should contain a real root, each of which gives the locations of the collinear points; a case where we can only think of a positive real root for each polynomial which gives three locations of the collinear points, respectively. However, it is pertinent to use numerical means to support the analytical efforts. Our numerical illustrations have been performed using the software *mathematica*, and we have taken $\mu = 0.01$ for the mass parameter, 0.99992 for radiation pressure factor of the bigger primary while for the heterogeneous spheroid we have used $k_2 = 1.58302 \times 10^{-7}$ and $k_3 = 3.13153 \times 10^{-8}$. Below in Table1, we compute the locations of roots with the help of the polynomials (9), (10) and (11) for $0 \le A_3 \le 0.06$ as shown below. It can be seen that increasing oblateness of the infinitesimal body, the collinear position L_1 moves away from the heterogeneous oblate spheroid while the same occur for the point L_3 which drifts away to the right of the radiating bigger primary.

From the Table 1, it is seen that there can be up to five collinear equilibrium points for $0 \le A_3 \le 0.04$ in the presence of radiation pressure, mass ratio and heterogeneity of the smaller primary.

Next, using equations (19), we numerically compute in Table 2 the positions of triangular points L_4 and L_5 which are defined by the mass parameter, radiation pressure of the bigger primary, heterogeneity of the smaller and oblateness of the infinitesimal mass. So we also take same parametric values as above with $0 \le A_3 \le 0.06$. For convenience and clarity, in Table 2, we also compute numerically the critical mass parameter given in equation (27).

Clearly form Table 2, as oblateness of the infinitesimal mass increases under the perturbing force of radiation and heterogeneous spheroid, the infinitesimal mass moves further away from the primaries and the critical mass decreases, which shows that the region of stable motion is decreasing.

Table 1. Collinear equilibrium points for $0 \le A_3 \le 0.06$ when $\mu = 0.01$, $q = 0.99992$, $k_2 = 1.58302 \times 10^{-7}$ and $k_3 = 3.13153 \times 10^{-8}$

A_3	L_1	L_2	L_3
0	-1.14681	-0.257278 0.24031 0.98997	0.937204
0.00001	-1.14684	-0.257256 0.240282 0.98998	0.937208
0.00015	-1.14732	-0.256945 0.239886 0.99005	0.937263
0.0005	-1.14847	-0.256165 0.238894 0.99023	0.937403
0.001	-1.15005	-0.255044 0.23747 0.99049	0.937602
0.015	-1.17775	-0.219904 0.1943 0.997618	0.937609
0.02	-1.18427	-0.205102 0.176857 1.00009	0.94501
0.04	-1.20393	-0.120451 0.0828447 1.00962	0.952496
0.05	-1.21155	1.0142	0.956129
0.06	-1.21828	1.01865	0.959692

Table 2. Position of triangular points and critical mass for $0 \le A_3 \le 0.06$ when $\mu = 0.01$, $q = 0.99992$, $k_2 = 1.58302 \times 10^{-7}$ and $k_3 = 3.13153 \times 10^{-8}$

A_3	x	$\pm y$	μ_C
0	-0.489965	0.866005	0.03852
0.00001		0.866011	0.0385171
0.00015		0.866092	0.0384759
0.0005		0.866294	0.0383729
0.001		0.866583	0.0382258
0.015		0.874666	0.0341059
0.02		0.877552	0.0326345
0.05		0.894873	0.0238062
0.06		0.900646	0.0208634

Table 3. Equilibrium points and stability results for $\mu = 0.01$, $q = 0.99992$, $k_2 = 1.58302 \times 10^{-7}$, $k_3 = 3.13153 \times 10^{-8}$ and $A_3 = 0.02$

Equilibrium points	Locations (x)	Locations (y)	μ_C	Characteristic Roots	Nature of motion
L_1	-1.18427	0	-	$\pm$1.01533 $\pm$2.15622i	Unstable
L_2	0.176857	0	-	$\pm$31.5959 & $\pm$9.91466i	Unstable
L_{21}	-0.205102	0	-	$\pm$7.8331 & $\pm$18.8756i	Unstable
L_{22}	1.00009	0	-	$\pm$0.391167 & $\pm$1.01888i	Unstable
L_3	0.94501	0	-	$\pm$0.810737 & $\pm$1.14899i	Unstable
L_4	-0.489965	0.877552	0.0326345	$\pm$0.282409 i & $\pm$0.927494i	Stable
L_5	-0.489965	-0.877552	0.0326345	$\pm$0.282409 i & $\pm$0.927494i	Stable

Finally, in Table 3, we put together the positions of the possible seven equilibria, the corresponding roots in each case and we state form of motion around the equilibrium points. As seen in Table 3, the four roots for the collinear point L_1 are complex and the point is an unstable equilibrium point. For the point L_2, L_{21}, L_{22} and L_3 there exist two real roots having opposite signs and two imaginary roots. These points are all unstable due to a positive root. In the case of the triangular points $L_{4,5}$ we have $\mu < \mu_C$ and the four roots are distinct pure imaginary. The triangular points are stable.

6. CONCLUSION

In this paper, equilibrium points and stability in the photogravitational restricted three-body problem with oblateness under a heterogeneous spheroid, have been examined when the bigger primary is a radiating body and the smaller one is a heterogeneous oblate spheroid having three layers with different densities while the infinitesimal mass is an oblate spheroid. The equilibrium points are found and their linear stability analyzed. It is seen that under some conditions there can be five collinear equilibrium points. These points are unstable either owning to a complex or positive root. Two triangular points exists and are stable provided $0 < \mu < \mu_c$. However, since μ_c is a decreasing function of the radiation pressure of the bigger primary, heterogeneous nature of the smaller primary and oblateness of the infinitesimal mass, we conclude that the nature of the bodies have destabilizing effect and the overall effect is that the region of stability around the triangular points is decreasing.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Lagrange J. Collected works of Lagrange. Panekoucke, Pari. 1772;9.
2. Szebehely VG. Theory of orbits, academic press, New York; 1967.
3. Singh J, Leke O. Stability of the photo-gravitational restricted three-body problem with variable masses. Astrophysics and Space Science. 2010;326:305.
4. Jain M, Aggarwal R. The restricted three body problem with Stokes drag effect. International Journal of Astronomy and Astrophysics. 2015;5:95.
5. Kumari S, Ansari AA, Farooq A. Existence and stability of L4 of the r3bp when the smaller primary is a heterogeneous triaxial rigid body with n layers. J. Appl. Environ. Biol. Sci. 2016;6:249.
6. Idrisi MJ. A study of libration points in CR3BP under albedo effect. International Journal of Advanced Astronomy. 2017;5: 6.
7. Ansari AA, Ziyad AA, Sadanand P. Circular restricted three-body problem when both primaries are heterogeneous spheroid of three layers and infinitesimal body varies its mass. Journal of Astrophysics and Astronomy. 2018;39:5.
8. Singh J, Amuda T.O. Stability analysis of triangular equilibrium points in the restricted three-body problem under effects of circumbinary disc, radiation and drag forces. Journal of Astrophysics and Astronomy. 2019;40:5.
9. Selim HH, Guirao JLG, Abouelmagd EI. Libration points in the restricted three–body problem: Euler angles, existence and stability. Discrete and Continuous Dynamical Systems Series S. 2019;12: 703
10. Kumar P, Sharma RK. Effect of radiation pressure on resonant periodic orbits in photo gravitational restricted three-body problem. International Journal of Mechanical and Production Engineering Research and Development (IJMPERD). 2020;10:1167.
11. Singh J, Leke O. Motion in a modified Chermnykh's R3BP with oblateness. Astrophysics and Space Science. 2014;350:143.
12. Abouelmagd EI, Juan LG. Guirao, On the perturbed restricted three-body problem. Applied Mathematics and Nonlinear Sciences. 2016;1:118.
13. Suraj MS, Hassan MR, Asique MC. The Photo gravitational R3BP when the Primaries are Heterogeneous Spheroid with Three Layers. American Astronautical Society. 2014;61:133.

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